Quantum particle in triangular well

In this note you'll see a naive attempt to solve the problem of a 1D quantum particle in the potential well of the form

$$V(x) = F\,|\,x\,|$$

The main problem as always is to find the energy eigenstates, which leads to the schroedinger's time independent equation

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + F\,|\,x\,|\,\psi = E\psi$$

The symmetry of the potential allows us to consider symmetric (even) or anti-symmetric (odd) state functions. So let's focus on the x>0 region only and solve the equation there. Having done that, building the complete answer won't be difficult.

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + Fx\psi = E\psi$$

Since the potential is non-negative everywhere, energy eigenvalue "E" can't be negative. So the two following real and non-negative quantities can be defined

$$k = \frac{\sqrt{2mE}}{\hbar}$$

$$\varepsilon = \frac{\hbar^2 k^3}{2mF}$$

$$u = kx$$

Simplifying the equation gives

$$\frac{d^2\psi}{du^2} = (\frac{u}{\varepsilon} - 1)\psi$$

Ofcourse there are so many different ways to solve the equation numerically. But the thing we are usually interested in is to answer the question "how the energy eigenvalues are distributed?". And that gives us the courage to try and find a series expansion for the answer and convert the differential equation to a difference equation which can give us the proper values of energy. (as is the case for hydrogen atom and the quantum SHO). Having checked the behavior of the equation at large values of u we can guess the answer to be of the form

$$\psi(u) = v(u)e^{-\frac{2}{3\sqrt{\varepsilon}}u^{\frac{3}{2}}}$$

In which v is a function which can be described by a finite power series expansion. Deriving the difference equation is a little cumbersome and I'm not going to do that here. But the important fact is that it fails to reveal energy eigenvalues! 1st because it's a 3rd degree difference equation and finding the eigenvalues turns out to be almost impossible. 2nd and the more

important reason is that you need to include half integer powers in the series too. These problems encourage us to use the "wag tail" method.

The basic idea is to find the values of epsilon which correspond to normalizable eigenfunctions by solving the equation numerically. Using MATLAB and Runge-Kutta method we can simply get to the following table of *proper* values of epsilon

odd	even
3.5752	1.0283
8.2653	5.8541
12.9715	10.5822
17.6803	15.3010
22.3908	20.0168
27.1020	24.7312
31.8135	29.4450
36.5253	34.1584
41.2372	38.8716
45.9492	43.5846
50.6613	48.297
55.3734	53.0103

Now if you write these values ordered you get

n	$\varepsilon(n)$
1	1.0283
2	3.5752
3	5.8541
4	8.2653
5	10.5822

6	12.9710
7	15.3010
8	17.6803
9	20.0168
10	22.3908
11	24.7312
12	27.1020
13	29.4450
14	31.8135
15	34.1584
16	36.5253
17	38.8716
18	41.2372
19	43.5746
20	45.9492
21	48.2975
22	50.6613
23	53.0103
24	55.3734

Having taken a glance at the table above we can guess that epsilon is growing linearly with n. the linear regression shows that the guess was not a bad one indeed.

$$r \approx 0.999998$$

$$\Rightarrow \varepsilon(n) = A + Bn$$

$$B = 2.358 \pm 0.001$$

$$A = -1.20 \pm 0.01$$

By translating the results in energy language we have

$$E(n) = [\frac{F\hbar}{\sqrt{2m}}(A + Bn)]^{2/3}$$

Which almost solves the problem. Using the relation above you can find the expected value of energy at a given temperature!