A Derivation of Kolmogorov's 4/5 Law

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1 Introduction

In the years 1939-41 Kolmogorov designated a theory of turbulent flows, (solutions of the incompressible Navier-Stokes equations in the high Reynolds number regime) and derived quantitative statistical results (The only quantitative results up to now) based on a minimal set of hypotheses. The following will include a derivation of Kolmogorov's four fifths law (The above mentioned quantitative result).

There were objections to Kolmogorov's hypotheses in the years after mainly put forward by Landau; which we shall not go through. Although it is worth mentioning that in 1962 the corrected picture was achieved by Kolmogorov-Obukhov known as KO62.

$$\partial_t v_i + v_j \partial_j v_i + \nabla p - \nu \nabla^2 v_i = f_i$$
$$\partial_i v_i = 0 \quad \textbf{or} \quad [\partial_i, v_i] = 0$$

Navier-Stokes equations for incompressible flows [1]

2 Scale by scale energy budget diffusion

The energy of the fluid is given by

$$E = \frac{1}{2} \int v_i v_i d^3\vec{r}$$

The time derivative of the total energy is

$$\frac{dE}{dt} = \int v_i \frac{\partial v_i}{\partial t} d^3\vec{r}$$

[1]What we denote by p in what follows or call 'pressure' is really the physical pressure normalized by ρ. After all one could always pick his favourite unit systems such that $\rho = 1$ holds if he wants.

$$= \int v_i(-v_j\partial_j v_i - \partial_i p + \nu\partial_j\partial_j v_i + f_i)d^3\vec{r}$$

using the incompressibility condition $[\partial_i, v_i] = 0$ we get

$$\frac{dE}{dt} = \int [-\frac{1}{2}\partial_j v_i v_i v_j - \partial_i(pv_i) + \nu\partial_j(v_i\partial_j v_i) - \nu(\partial_j v_i)(\partial_j v_i) + f_i v_i]d^3\vec{r}$$

if the energy is to be finite, velocity field must vanish at $r \to \infty$ therefore the derivatives don't contribute and we get

$$\frac{dE}{dt} = \int [-\nu(\partial_j v_i)(\partial_j v_i) + f_i v_i]d^3\vec{r} \tag{1}$$

if we define the vorticity vector $\omega_k := \varepsilon_{ijk}\partial_i v_j$

the last equation can be rewritten in terms of $\omega_i\omega_i$

$$\omega_k\omega_k = (\varepsilon_{ijk}\partial_i v_j)(\varepsilon_{mnk}\partial_m v_n)$$

$$= (\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm})(\partial_i v_j)(\partial_m v_n)$$

$$= (\partial_i v_j)(\partial_i v_j) - (\partial_i v_j)(\partial_j v_i)$$

$$= (\partial_i v_j)(\partial_i v_j) - \partial_i\partial_j v_i v_j$$

the second term is a derivative and hence will disappear via integration. Therefore

$$\frac{dE}{dt} = \int [-\nu\omega^2 + f_i v_i]d^3\vec{r} \tag{2}$$

It will prove useful to work with the Fourier transforms from time to time. We denote the Fourier transform of a quantity $q(\vec{r})$ by $Q(\vec{k})$ as follows

$$Q(\vec{k}) = \frac{1}{\sqrt{2\pi}}\int q(\vec{r})e^{-i\vec{k}.\vec{r}}d^3\vec{r}$$

$$\vec{V} = \frac{1}{\sqrt{2\pi}}\int \vec{v}(\vec{r})e^{-i\vec{k}.\vec{r}}d^3\vec{r} \tag{3}$$

Parseval's theorem gives

$$E = \frac{1}{2}\int V_i^* V_i d^3\vec{k}$$

and

$$\frac{dE}{dt} = -\nu\int \Omega_i^*\Omega_i d^3\vec{k}$$

$$= -\nu\int |i\vec{k}\times\vec{V}|^2 d^3\vec{k}$$

$$= -\nu\int \varepsilon_{ijk}\varepsilon_{mnk}k_i V_j^* k_m V_n d^3\vec{k}$$

incompressibility $\vec{k}.\vec{r} = 0$ gives

$$\frac{dE}{dt} = -\nu \int k^2 |V|^2 d^3\vec{k}$$

As can be seen there is no contribution to *dissipation* from the nonlinear terms. Although if we define E_K as follows

$$E_K = \int_{|\vec{k}|<K} |V(\vec{k})|^2 d^3\vec{k}$$

To be the energy content in the wavenumbers less than k, there will be a flow of energy between different wave numbers due to nonlinearities

$$\frac{dE_K}{dt} = \mathrm{Re}[\int_{|\vec{k}|<K} V_i^* \partial_t V_i d^3\vec{k}\ ]$$

$$\partial_t V_i = \{\mathbf{N.L.}\} - ik_i P - \nu k^2 V_i + F_i$$

in which $\{\mathbf{N.L.}\}$ denotes the nonlinear term's contribution. Substituting the result we get

$$\frac{dE_K}{dt} = \{\mathbf{N.L.}\} + \mathrm{Re}\{\int_{|\vec{k}|<K} V_i^*(-ik_i P - \nu k^2 V_i + F_i) d^3\vec{k}\ \}$$

Since the velocity field is real, we have $V^*(\vec{k}) = V(-\vec{k})$. This and the incompressibility condition guarantee that the first term in the integral vanishes. The second term is the dissipation rate in wavenumbers less than K and the last one the external work done by external forces in wave numbers less than K. [2]

Now to calculate the nonlinear term, it is easier to compute this term in the physical space rather than Fourier space. We use the following notation: For any field quantity q, the symbol $q_K^<$ ($q_K^>$) denotes the output of the ideal lowpass (highpass) filter with cutoff frequency K and input q. it is clear that $q = q_K^< + q_K^>$

$$\frac{dE_K}{dt}|_{\{\mathbf{N.L.}\}} = -\int d^3\vec{r}\ \vec{v}_K^<(\vec{r}).\{\vec{v}.\nabla\vec{v}\}_K^<$$
$$= -\int d^3\vec{r}\ \vec{v}_K^<(\vec{r})\vec{v}.\nabla(\vec{v}_K^< + \vec{v}_K^>)$$

[2]The definition of the two recent quantities is a little different than usual; to compute the energy (work) content in wavenumbers less than K one does *not* calculate the Fourier transform of the local energy (work) instead acts a low pass filter on the first order fields i.e. velocity, force, etc. and computes the total energy content (work) using the filtered fields.

The first term vanishes.[3] Defining

$$\Pi_K := \int v^{<}_{K_i} v_j \partial_j v^{>}_{K_i} d^3\vec{r}$$

We get the desired *scale by scale energy budget diffusion equation*

$$\frac{dE_K}{dt} + \Pi_K = D_K + W_K \tag{4}$$

in which the last two terms represent the dissipation rate and external injected power respectively.

3 Symmetries of Navier-Stokes equations

Looking at Navier-Stkes homogenous equations there are some *transformations* under which a solution remains a solution, these *symmetries* will help us derive relations in turbulent flows where the boundary conditions (boundary conditions obviously break these symmetries) are of less significance.

$$\partial_t v_i + v_j \partial_j v_i + \partial_i p - \nu \partial_j \partial_j v_i = 0 \tag{5}$$

It is worth noting that p in the equation is not an independent variable and is itself a function of the velocity field v. Although this functionality is intrinsic in the equation and may be derived by taking both curl and divergences from Navier-Stokes; hence we will not discuss transformations on p and simply ignore the variance/invariance of the corresponding term.

3.1 Space-Time translation

It is clear that for a solution $v_i(\vec{r}, t)$, the transformed field $v_i(\vec{r} - \vec{d}, t - \tau)$ is also a solution.

3.2 Space rotation

Since there is no preferred space direction implicit in the Navier-Stokes, it is evident that the transformed solution

$$R^{-1}\vec{v}(R\vec{r}, t)$$

for any rotation matrix R is also a solution.

[3] for any divergenceless vector v_i

$$\int u_i v_j \partial_j u_i dV = \frac{1}{2} \int \partial_j v_j u_i u_i dV = 0$$

3.3 A special scaling

Let us re-scale space and time with two different sacalars and ask the solutions to remain solutions.

$$\vec{r} \to \lambda \vec{r}$$

$$t \to \lambda^{1-h} t$$

and naturally

$$v_i \to \lambda^h v_i$$

For the Navier-Stokes to remain valid, every term should be multiplied by a common factor. This condition is only met if

$$h = -1$$

And hence a single group of transformations is present corresponding to $h = -1$ re-scaling of space-time.

3.4 Infinite Re, infinite symmetries

In Kolmogorov's 1941 papers describing turbulent flows, there is a *postulate* stating that in the limit of infinite Reynolds number (equivalently $\nu \to 0$) the scaling symmetry spreads to every h (not only $h = -1$) hence resulting in an infinite number of scaling symmetry classes (coresponding to different h values).

4 Probabilistic interpretation through random processes

4.1 Why a probabilistic description

Providing statistic description of deterministic dynamical systems is useful for 'complicated enough' systems, like those studied in the window of (classical) statistical physics with too many variables or systems with chaotic (not to be defined carefully here) dynamics. Using statistical quantities is legitimate in the first example simply because we don't need (or are not able) to track the evolution of an enormous number of dynamical variables and hence focus on statistical variables i.e. mean-values. While in the second example randomness rises from small (but undeniable) uncertainties in the initial state of system since two close initial conditions can exhibit significant differences in their future behaviour.

A very simple example of a chaotic dynamical system is the following one-dimensional discrete time system:

$$x[t+1] = 1 - 2\left|x[t] - \frac{1}{2}\right| \qquad x \in [0, 1]$$

Let's say we measure the initial value $x[0]$ with some uncertainty. The measurement results in a probability density function for $x[0]$ which we denote by $f_0(x)$. A typical initial distribution is a gaussian like spike near some value μ with some finite width. The reader can convince himself that the evolution of distribution functions is given by

$$f_{t+1}(x) = \frac{1}{2}\Big[f_t(\frac{x}{2}) + f_t(\frac{1-x}{2})\Big]$$

Or in Fourier series language

$$a[t+1, n] = a[t, 2n]$$

Where $a[t, n]$'s are Fourier components of $f_t(x)$

$$f_t(x) = \sum_{n=0}^{\infty} a[t, n] \cos(n\pi x)$$

Looking at Fourier components evolution, it is clear that no matter how narrow the initial value is measured, after some iterations all frequencies but the DC value tend to zero resulting in a unifom distribution.

This monotone loss of information and convergence to a distribution (uniform in this case) leaves us no choice but to study the statistical characteristics of the limiting distribution.

4.2 Random processes and correlation functions

A random process is defined as a random *signal* in time (or space-time) in other words it is a set of random variables labelled with space-time co-ordinates $x(\vec{r}, t)$.

The turbulent flow will be regarded as a random process. Both homogeneity and isotropy will be assumed which means the distribution of random variables $x(\vec{r}, t)$ are independent of the space-time labels. Isotropy also implies that the distributions for velocity are centered.

$$\langle v_i \rangle = 0$$

We will be interested in the correlation functions which we'll call structure-functions to describe the turbulent flow.

$$T^{\{i\}}(\vec{l}) = \langle \delta v_{i_1}(\vec{l}) \delta v_{i_2}(\vec{l}) \cdots \delta v_{i_p}(\vec{l}) \rangle$$

in which $\{i\}$ denotes the set of indices and

$$\delta v_i(\vec{l}) := v_i(\vec{r} + \vec{l}, t) - v_i(\vec{r}, t)$$

independent of $\{\vec{r}, t\}$ because of space-time homogeneity.

A simpler structure function is the longitudinal structure function

$$S^{(p)}(\vec{l}) = \langle (\delta \vec{v}(\vec{l}).\hat{\vec{l}})^p \rangle$$

or simply

$$S^{(p)} = \langle \delta v_{||}^{p} \rangle$$

Which are only functions of $l = |\vec{l}|$ only because of isotropy.

It is worth noting in the end that the homogeneity in space-time and the fact that correlation functions go to zero for large increments in space-time makes the possibly different time averging, space averaging and ensemble averaging equivalent. We simply denote all of them by $\langle\rangle$ symbols.

4.3 Energy distribution

Consider an n diensional homogenous and isotropic random process x (a random signal in n dimensions). We will show the space-time coordinates with the n dimensional vector $\vec{t}$. Desired is the distribution of the energy content of the signal x in different frequencies.

Consider a signal $w(\vec{t})$ with corresponding Fourier transform

$$W(\vec{\omega}) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int e^{-i\vec{\omega}.\vec{t}} w(\vec{t}) d^n\vec{t}$$

We wish to compute the DC value or time average of w given it's Fourier transform W. It is clear that the DC value will be a function of the near zero behavior of W only. Since the Fourier transform is linear we need only to examine the Fourier transform of a pure DC signal $w(\vec{t}) = 1$

$$W(\vec{\omega}) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int e^{-i\vec{\omega}.\vec{t}} 1 d^n\vec{t}$$

$$= (2\pi)^{\frac{n}{2}} \delta(\vec{\omega})$$

hence

$$\langle w \rangle = \frac{1}{(2\pi)^{\frac{n}{2}}} \lim_{\Omega\to 0} \int_{|\vec{\omega}|<\Omega} W(\vec{\omega}) d^n\vec{\omega}$$

An interesting DC value is the 2-point correlation function

$$g(\vec{s}) := \langle x(\vec{t}) x(\vec{t} + \vec{s}) \rangle$$

The Fourier transform of the product is simply given by the convolution integral

$$F\{x(\vec{t})y(\vec{t})\} = \frac{1}{(2\pi)^{\frac{n}{2}}} X(\vec{.}) * Y(\vec{.})$$

where X and Y are the proper Fourrier transforms. Hence

$$F\{x(\vec{t})x(\vec{t} + \vec{s})\} = \frac{1}{(2\pi)^{\frac{n}{2}}} X(\vec{.}) * [X(\vec{.}) exp(i\vec{.}.\vec{s})]$$

$$= \frac{1}{(2\pi)^{\frac{n}{2}}} \int d^n\vec{\omega}' X(\vec{\omega}') X(\vec{\omega} - \vec{\omega}') e^{i(\vec{\omega}-\vec{\omega}').\vec{s}}$$

therefore

$$g(\vec{s}) = \frac{1}{(2\pi)^n} \lim_{\Omega\to 0} \int_{|\vec{\omega}|<\Omega} d^n\vec{\omega} \int d^n\vec{\omega}' X(\vec{\omega}')X(\vec{\omega}-\vec{\omega}')e^{i(\vec{\omega}-\vec{\omega}').\vec{s}}$$

Keeping in mind the identity $X^*(\vec{\omega}) = X(-\vec{\omega})$ we get

$$g(\vec{s}) = (2\pi)^{\frac{n}{2}} F^{-1}\{|X(\omega)|^2\} \tag{6}$$

Which is known as Wiener-Khinchin-Einstein formula. This simply states that the energy distribution function of the random process is proportional to the Fourier transform of the 2-point correlation function.

Isotropy and homogeneity also give

$$g(\vec{s}) = g(s)$$

$$|X(\vec{\omega})|^2 = u(\omega)$$

The Wiener-Khinchin-Einstein formula reduces to

$$g(s) = \frac{2\pi^{(\frac{n-1}{2})}}{\Gamma(\frac{n-1}{2})} \int_0^\infty d\omega\omega^{n-1}u(\omega) \int_0^\pi cos(\omega s cos(\theta)) sin^{n-2}(\theta) d\theta \tag{7}$$

for $n = 3$ (3 dimensional space)

$$g(s) = \frac{4\pi}{s} \int_0^\infty \omega u(\omega) sin(\omega s) d\omega \tag{8}$$

Of interest is the special case in which energy distribution follows a power law.

$$u(\omega) = A\omega^{-\alpha-n+1} \qquad A > 0$$ [4]

Although no such energy distribution results in a finite total energy, for $\alpha \in (1,3)$ the structure function $\langle \delta x(s)^2 \rangle$ remains finite. For $n = 3$ and power law energy distribution:

$$\langle \delta x(s)^2 \rangle \propto \omega^{\alpha-1} \tag{9}$$

As we promised we shall seek a statistical interpretation of observable physical quantities and rules governing them in turbulent flow regimes. Equipped with concepts of stochastic processes we can consider the velocity field $\vec{v}$ to be a stochastic process through space and time. The statistical properties are assumed *stationary* with respect to space-time co-ordinates.

5 Experimental facts of fully developed turbulence

In this section we'll meet two experimental laws observed in turbulent flows.

[4] The tedious looking power is chosen such that the energy density is proportional to $\omega^{-\alpha}$. The degeneracy of ω will correct the power.

5.1 Two thirds law

The quantity $\langle \delta v(l)^2 \rangle$ has been measured frequently in different labs and the proportionality

$$\langle \delta v(l)^2 \rangle \propto l^{2/3}$$

has been observed. Comparing with eq. (9) we get

$$\alpha = \frac{5}{3}$$

which is also in good agreement with experimental data. This is called the two thirds power law.

5.2 Dissipative anomaly

As we derived in the 2nd section it is clear that for non-viscous flows, energy is conserved. It is also evident from dimensional analysis that flows with the same Reynolds number should behave in a similar way hence it may be stated that in the limit of infinite Reynolds numbers, energy is also conserved. The experimental data suggest otherwise: if we denote the dissipation rate per mass with ε it is observed from experiments that

$$\left(\lim_{Re \to \infty} \varepsilon \right) > 0 = \varepsilon(\infty)$$

i.e. the dissipation rate tends to a finite non-zero value as the Reynolds number is increased to infinity.

It may be regarded as a consequence of self similarity of the flow since for self similar objects, a unique length-scale and hence a unique Reynolds number is not well defined.

6 K41[5]

In his third 1941 turbulence paper Kolmogorov found that an *exact* relation can be derived for the third order longitudinal structure function, the average of the cube of the longitudinal velocity increment. He assumed homogeneity, isotropy and an additional hypothesis about the finiteness of the energy dissipation. Without any further assumptions he derived the following result from the Navier-Stokes equation:

Four fifths law. *In the limit of infinite Reynolds number, the third order (longitudinal) structure function of homogenous isotropic turbulence, evaluated for increments l small compared to the coherency length, is given in terms of the mean energy dissipation per unit mass ε (assumed to remain finite and nonvanishing) by*

$$\langle \delta v_{||}^3(l) \rangle = -\frac{4}{5}\varepsilon l \tag{10}$$

[5] This subsection is quoted from the refrence.

6.1 Karman-Howarth-Monin relation

Let's start by taking the dot product of the velocity in two different points $\vec{r}$ and $\vec{r}+\vec{l}$ namely $\vec{v},\ \vec{v}'$. For the average of the dot product we have

$$\frac{1}{2}\partial_t\langle\vec{v}.\vec{v}'\rangle$$

$$= \langle(\partial_t v_i)v_i' + v_i(\partial_t v_i')\rangle$$

$$= -\frac{1}{2}\langle(\partial_j v_i v_j)v_i'\rangle - \frac{1}{2}\langle v_i(\partial_j v_i' v_j')\rangle$$

$$-\frac{1}{2}\langle v_i'\partial_i p\rangle - \frac{1}{2}\langle v_i\partial_i p'\rangle$$

$$+\frac{1}{2}\langle f_i v_i'\rangle + \frac{1}{2}\langle v_i f_i'\rangle$$

$$+\frac{1}{2}\nu\langle(\partial_j\partial_j v_i)v_i'\rangle + \frac{1}{2}\nu\langle v_i(\partial_j\partial_j v_i')\rangle$$

in which all the derivatives are taken in constant $\vec{l}$. The pressure terms are zero according to incompressibility. The external force terms can be written as

$$\frac{1}{2}\langle\vec{f}(\vec{r}).\vec{v}(\vec{r}+\vec{l}) + \vec{v}(\vec{r}).\vec{f}(\vec{r}+\vec{l})\rangle$$

$$= \frac{1}{2}\langle\vec{f}(\vec{r}-\vec{l}).\vec{v}(\vec{r}) + \vec{v}(\vec{r}).\vec{f}(\vec{r}+\vec{l})\rangle$$

$$= \langle\vec{v}(\vec{r}).\Big(\frac{\vec{f}(\vec{r}-\vec{l})+\vec{f}(\vec{r}+\vec{l})}{2}\Big)\rangle$$

Where use of homogeneity has been made in the derivation.

The viscous terms can also get simplified using homogeneity. If we change the independent variables from $(\vec{r},\vec{l})$ to $(\vec{r},\vec{r}'=\vec{r}+\vec{l})$ then the viscous contribution becomes

$$+\frac{1}{2}\nu(\partial_j\partial_j + \partial_j'\partial_j')\langle v_i v_i'\rangle$$

Back to variables$(\vec{r},\vec{l})$ the partial derivatives will take the form

$$\partial_i \to \partial_i - \partial_i^l \quad , \quad \partial_i' \to \partial_i^l$$

and the contribution becomes

$$\nu\nabla_l^2\langle v_i v_i'\rangle$$

Next, we observe that

$$\langle|\delta v|^2\delta\vec{v}\rangle = -\langle {v'}^2\vec{v}\rangle + \langle v^2\vec{v}'\rangle - 2\langle\vec{v}.\vec{v}'\delta\vec{v}\rangle$$

(All the additional terms cancel by isotropy) Taking divergence in the $\vec{l}$ space of the last quantity we get

$$-\frac{1}{4}\nabla_l.\langle|\delta v|^2\delta\vec{v}\rangle = -\frac{1}{2}\partial_t\langle\vec{v}(\vec{r}).\vec{v}(\vec{r}+\vec{l})\rangle+\nu\nabla_l^2\langle\vec{v}(\vec{r}).\vec{v}(\vec{r}+\vec{l})\rangle+\langle\vec{v}(\vec{r}).\frac{\vec{f}(\vec{r}-\vec{l})+\vec{f}(\vec{r}+\vec{l})}{2}\rangle \tag{11}$$

Which is the desired Karman-Howarth-Monin relation. The quantity in eq. (12) will be called the physical space energy flux, since it is equal to

$$\varepsilon(\vec{l}) = -\frac{1}{2}\partial_t\langle\vec{v}(\vec{r}).\vec{v}(\vec{r}+\vec{l})\rangle\Big|_{\{\mathbf{N.L.}\}} \tag{12}$$

Or the nonlinear contribution to the time derivative.

Note that for very small increments $\vec{l}$ the divergence term vanishes (assuming differentiability of the velocity field) and we get

$$\partial_t\frac{1}{2}\langle v^2\rangle = \langle\vec{f}.\vec{v}\rangle + \nu\langle\vec{v}.\nabla^2\vec{v}\rangle$$

which is essentially the same as eq. (1). It is also worth mentioning that all time derivatives will vanish in the case of a stationary turbulence.

6.2 An expression for energy flux

Back in section 2 we defined the quantity Π_K to be minus the nonlinear energy flux in wave numbers less than K. Hence Π_K can be written in terms of energy flux we just derived to be

$$\Pi_K = \frac{1}{(2\pi)^3}\int_{|\vec{k}|<K} d^3\vec{k}\int d^3\vec{l}\, e^{-i\vec{k}.\vec{l}}\varepsilon(\vec{l})$$

Evaluating the integral over $\vec{k}$

$$\Pi_K = \frac{1}{2\pi^2}\int d^3\vec{l}\,\varepsilon(\vec{l})\,\frac{sin(Kl) - Kl\,cos(Kl)}{l^3}$$

Or equivalently (using integration by parts)

$$\Pi_K = \frac{1}{2\pi^2}\int d^3\vec{l}\,\frac{sin(Kl)}{Kl}\,\nabla_l.\Big(\varepsilon(\vec{l})\frac{\vec{l}}{l^2}\Big)$$

substituting ε (eq. (13)) we get

$$\Pi_K = -\frac{1}{8\pi^2}\int d^3\vec{l}\,\frac{sin(Kl)}{Kl}\,\nabla_l.\Big[\frac{\vec{l}}{l^2}\,\nabla_l.\langle\delta v^2\delta\vec{v}\rangle\Big] \tag{13}$$

Now to simplify more the relation for Π_K using isotropy statements, we begin by two tensor definitions

$$b_{ijm} := \langle v_i v_j v'_m\rangle$$

$$B_{ijm} := \langle (v'_i - v_i)(v'_j - v_j)(v'_m - v_m) \rangle$$

Both tensors should be expressible in term of isotropic tensors (Kronecker's delta, Levi-Civita epsilon) and tensors made of the unit vector $\hat{l} := \frac{\vec{l}}{l}$. It should also be symmetrical in the first two indices. The most general form is then

$$b_{ijm} = C(l)\delta_{ij}\hat{l}_m + D(l)(\delta_{im}\hat{l}_j + \delta_{jm}\hat{l}_i) + F(l)\hat{l}_i\hat{l}_j\hat{l}_m$$

incompressibility $\partial_m b_{ijm} = 0$, using the identities $\partial_i \hat{l}_j = \frac{\delta_{ij} - \hat{l}_i\hat{l}_j}{l}$ and $\partial_i l = \hat{l}_i$ reads

$$\delta_{ij}\Big(\frac{2(C+D)}{l} + C'\Big) + \hat{l}_i\hat{l}_j\Big(2D' - \frac{2D}{l} + F' + \frac{2F}{l}\Big) = 0$$

or

$$\frac{2(C+D)}{l} + C' = 0$$

$$2D' - \frac{2D}{l} + F' + \frac{2F}{l} = 0$$

The second equation can be simplified using the first as follows

$$0 = F' + 2D' + \frac{2F}{l} + \frac{4D}{l} - \frac{6D}{l}$$

$$= F' + 2D' + \frac{2F}{l} + \frac{4D}{l} + \frac{6}{l}\Big(\frac{l}{2}C' + C\Big)$$

$$= \Big(F + 2D + 3C\Big)' + \frac{2}{l}\Big(F + 2D + 3C\Big)$$

which results in

$$F + 2D + 3C = 0$$

for bounded solutions. Re-expressing D and F in terms of C, b becomes

$$b_{ijm} = C\delta_{ij}\hat{l}_m - (C + lC'/2)(\delta_{im}\hat{l}_j + \delta_{jm}\hat{l}_i) + (lC' - C)\hat{l}_i\hat{l}_j\hat{l}_m \tag{14}$$

And by using isotropy conditions B can also get simplified

$$B_{ijm} = -2(lC' + C)(\delta_{ij}\hat{l}_m + \delta im\hat{l}_j + \delta_{jm}\hat{l}_i) + 6(lC' - C)\hat{l}_i\hat{l}_j\hat{l}_m \tag{15}$$

Having established these preliminary results, we now observe that

$$S^{(3)}(l) = \langle \delta v_{||}^3 \rangle = B_{ijm}\hat{l}_i\hat{l}_j\hat{l}_m = -12C \tag{16}$$

We also expect from isotropy that the quantity $\langle \delta v^2 \delta\vec{v} \rangle$ is parallel to $\vec{l}$ and are hence interested only in $\hat{l}$ component

$$\hat{l}.\langle \delta v^2 \delta\vec{v} \rangle = B_{iim}\hat{l}_m = -4lC' - 16C \tag{17}$$

substituting back to eq. (14) reads

$$\Pi_K = -\frac{1}{6\pi}\int_0^\infty dl\frac{sin(Kl)}{l}(1 + l\frac{d}{dl})(3 + l\frac{d}{dl})(5 + l\frac{d}{dl})\frac{S^{(3)}(l)}{l} \tag{18}$$

6.3 The four fifths law

Now to derive the desired four fifths law we need to add a final assumption to the previous list.

The driving force $\vec{f}(t,\vec{r})$ acts only at large scales compared to the coherency length, $\ell_0 \sim K_c^{-1}$ in other words

$$\vec{f}_K^{<}(t,\vec{r}) \approx \vec{f}(t,\vec{r}), \qquad \textbf{for} \quad K > K_c$$

After the stationary state is reached, the time derivatives vanish. Energy conservation reads

$$\langle \vec{f}.\vec{v}\rangle = -\nu\langle \vec{v}.\nabla^2\vec{v}\rangle = \varepsilon$$

The energy in wavenumbers less than K is also conserved.(Cf. eq. (4))

$$\Pi_K = W_K + D_K$$

The first assumption, reads in this context

$$W_{K>K_c} \approx W_\infty = \langle \vec{f}.\vec{v}\rangle = \varepsilon$$

In the limit $\nu \to 0$ the quantity D_K goes to zero for any fixed K since the Reynolds number goes to zero for finite wave numbers and the finite non-zero energy dissipation will be transported to higher wavenumbers (where Reynolds number is significant).

$$\lim_{\nu\to 0} D_K = 0$$

hence

$$\Pi_K = W_K = \varepsilon \tag{19}$$

Eq. (18) now reads

$$\begin{aligned}\varepsilon &= -\lim_{l\to 0} J(l)\int_0^\infty dx\frac{sin(x)}{x} \\ &= -\frac{\pi}{2}\lim_{l\to 0} J(l)\end{aligned}$$

in which

$$J(l) := (1 + l\frac{d}{dl})(3 + l\frac{d}{dl})(5 + l\frac{d}{dl})\frac{S^{(3)}(l)}{6\pi l}$$

Assuming $S^{(3)}(l) \propto l^\beta$, we get

$$S^{(3)}(l) = \frac{-12\varepsilon l^\beta}{\beta(2+\beta)(4+\beta)}$$

If we wish to get a finite limit for J in the small scale limit (this sets ε to stay finite according to the third assumption), we must have

$$\beta = 1$$

Which leads to

$$\boxed{S^{(3)}(l) = -\frac{4}{5}\varepsilon l} \tag{20}$$

This completes the derivation of the four fifths law.

7 Refrence

U. Frisch, *Turbulence, The legacy of A. N. Kolmogorov* (Cambridge 1995)