Lamb Shift
A *semiclassical* survey

1 The Phenomenon

According to a first order perturbation theoretical view in fine structure constant, α, energy levels of a hydrogen atom with hamiltonian

$$H = \frac{p^2}{2m} - \frac{e^2}{x} - \frac{p^4}{8m^3} + \frac{e^2}{2m^2x^3}\mathbf{S}.\mathbf{L}^1$$

are given by

$$E_{nj} = -\frac{me^4}{2n^2}\Big[1 + \frac{e^4}{n^2}(\frac{n}{j+1/2} - \frac{3}{4})\Big]$$

and hence depend only on the principal quantum number n and total angular momentum j. This result (consistent also with Dirac's relativistic theory) hence predicts no energy difference between levels $2S_{1/2}$ and $2P_{1/2}$ (They both have $n = 2$, $j = 1/2$).

Lamb and Rutherford however discovered a shift (known as Lamb shift) between the two levels corresponding to a wavelength of about 30 cm.

To describe this, a first step in the right track was taken by H. A. Bethe using the classical quantum theory *attached* with vacuum fluctuations of electromagnetic fields in the same year (1947) with a reasonably good prediction of frequency. In what follows we pretty much follow Bethe's theory and give an example which this semiclassical theory can not describe properly.

2 A Theory

The classical quantum theory is unable to describe phenomena such as spontaneous radiation and Lamb shift unless electromagnetic fields are also treated quantum mechanically. An important resullt of quantising fields is that "vacuum ground state has non-zero energy and hence fluctuating EM fields!" Any alternative for the standard theory (known as QED) should hence include this fact.

Closest possible alternative to old quantum theory is therefore {QM+"vacuum fluctuates!"}. In other words matter will be treated quantum mechanically; electromagnetic fields however will obey Maxwell's equations[2] superposed with God-given *stochastic* fields due to vacuum ground state.

Stochastic properties of this zero-field are

$$\langle E_i^0 \rangle = \langle B_i^0 \rangle = 0$$

$$\langle E_i^0 E_i^0 \rangle = \langle B_i^0 B_i^0 \rangle = \frac{2}{\pi}\omega^3 d\omega$$

The constants are *adjusted* such that the ground state energy of a single mode with frequency ω will be $\omega/2$.

[1] In our units $4\pi\varepsilon_0 = \frac{\mu_0}{4\pi} = \hbar = G = k_B = 1$

[2] Sources will be given by density operators in a trivial fashion. This is not of interest in our discussion though.

Now consider a particle with mass m and charge e subject to a potential $V(\mathbf{x})$ and EM potentials $(\phi, \mathbf{A})$. The hamiltonian becomes

$$H = \varphi(\mathbf{x}) + \frac{1}{2m}(\nabla^2 + 2ie\mathbf{A}'.\nabla + e^2\mathbf{A}'^2)$$

With $\mathbf{A}' = \mathbf{A} + \mathbf{A}^0$ representing the superposition of original and fluctuating field. Perturbative Hamiltonian is

$$\delta H = \frac{e^2}{m}\mathbf{A}_0.\mathbf{A} - \frac{e}{m}\mathbf{A}_0.\mathbf{p} + \frac{e^2}{2m}A_0^2$$

This is in effect equivalent[3] to exerting a periodic force (due to fluctuating electric field) and making the electron oscillate. As a result, the electron $feels$ an average potential

$$V_{eff}(\mathbf{x}) = \langle V(\mathbf{x} + \mathbf{x}^{\text{osc.}})\rangle_{osc.}$$

For small oscillations

$$V_{eff}(\mathbf{x}) = \langle V(\mathbf{x}) + \mathbf{x}_i^{\text{osc.}}\partial_i V(\mathbf{x}) + \frac{1}{2}\mathbf{x}_i^{\text{osc.}}\mathbf{x}_j^{\text{osc.}}\partial_i\partial_j V(\mathbf{x}) + \cdots\rangle_{osc.}$$

The linear terms disappear (symmetry) and the first non-vanishing term will be

$$\frac{1}{3}(\nabla^2 V)\langle x^2_{\text{osc.}}\rangle_{\text{osc.}}$$

To estimate the oscillation amplitudes we use a free electron model.

$$ma = F_0 e^{i\omega t} \Rightarrow x = \frac{F_0}{m\omega^2}e^{i(\pi+\omega t)}$$

r.m.s amplitudes for different frequencies add up to yield

$$\langle x^2_{\text{osc.}}\rangle = \frac{1}{2m^2}\sum_{\omega}\frac{F_0^2(\omega)}{\omega^4}$$

$$= \frac{e^2}{2m^2}\int\frac{I(\omega)}{\omega^4}d\omega = \frac{e^2}{\pi m^2}\log\left(\frac{\omega_{max}}{\omega_{min}}\right)$$

In very low frequencies, comparable to hydrogen frequency, our electron is no longer $free$. This means we need write for lower limit

$$\omega_{min} = me^4$$

In very high frequencies (comparable to m), our electron is no longer *single*. This means

[3] A perhaps *neater* way would be to avoid using the concept of force and treat a stochastic perturbative Hamiltonian quantum mechanically. The interpretation of *stochastic* here needs to be adjusted in a way to obtain the same results.

$$\omega_{max} = m$$

and

$$\frac{\omega_{max}}{\omega_{min}} = e^{-4} = \alpha^{-2}$$

The perturbation becomes

$$\delta H = \frac{e^2}{3\pi m^2} \log\left(\frac{\omega_{max}}{\omega_{min}}\right) \nabla^2 V$$

In the case of hydrogen, $V = -e^2/x$ and

$$\delta H = \frac{-4e^4}{3m^2} \log(e^4) \delta^3(\mathbf{x})$$

Now this term makes for energy difference between $2^2 S_{1/2}$ and $2^2 P_{1/2}$. The difference being (using $\psi_{100}(0) = 1/\sqrt{16\pi a^3}$)

$$\Delta E_{\text{Lamb}} = E(2^2 S_{1/2}) - E(2^2 P_{1/2}) = -\frac{m}{6\pi} \alpha^5 \log(\alpha)$$

with $\alpha \equiv e^2$ being the famous fine structure constant. This predicts the frequency to be about 670MHz which is far from being exact but is of the same order and serves reasonably well as a first estimation. Note that the observed frequency is 1057 MHz.

3 Flaws (Qubeats!)

How far can this *mixed* theory take us? Is there any phenomenon which this theory can not describe? The answer is affirmative indeed; the example is called "Quantum beat phenomenon" and is observed in laboratories confirming QED predictions.

Consider a three level system prepared in the state

$$|\psi\rangle = c_a e^{-i\omega_a t} |a\rangle + c_b e^{-i\omega_b t} |b\rangle + c_c e^{-i\omega_c t} |c\rangle$$

$E_a \gg E_c > E_b = 0$. Further assume that the dipole operator $\mathcal{P}$ is such that transitions from $|a\rangle$ to both $|b\rangle$ and $|c\rangle$ are allowed. Our semiclassical theory predicts

$$\mathbf{p}(t) = \mathcal{P}_{ac} c_a^* c_c e^{i\omega_{ac} t} + \mathcal{P}_{ab} c_a^* c_b e^{i\omega_{ab} t} + \text{c.c}$$

The oscillating field generated will exhibit a beats phenomenon due to close frequencies of the two possible transitions.

QED however says

$$|\psi(t)\rangle = c_a e^{-i\omega_a t} |a, 0\rangle + c_b e^{-i\omega_b t} |b, 0\rangle + c_c e^{-i\omega_c t} |c, 0\rangle + T_{ab}(t) |b, 1_{\omega_{ab}}\rangle + T_{ac}(t) |c, 1_{\omega_{ac}}\rangle$$

In which the second state, represents EM field and T's are transition probability amplitudes as functions of time. The electric field is the creation operator

$$E_\omega(t) \propto a_\omega$$

And the interfering term becomes

$$\langle E_{\omega_{ac}} E_{\omega_{ab}} \rangle \propto \langle c|b \rangle = 0$$

And no interference (beats) is predicted. This is in accordance with experimental observations ofcourse.

4 References

1. *Electromagnetic shift of energy levels*, H. A. Bethe, 1947
1. Marlan O. Scully, Suhail Zubairy, *Quantum optics*, Cambridge University press, 1997