A Proof for Behzad's Conjecture

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If you have a flat piece of paper around, you certainly can make a paper cylinder by folding it. You can also make a paper cone but it should not take you long to realize that the same is not true for a sphere. What is the difference between a cylinder and a sphere? What surfaces can you make with a piece of paper? Asking a general relativist or a differential geometrist one gets the sophisticated answer: "The surface needs to be intrinsically flat, or equivalently the Riemann curvature tensor must vanish everywhere."

As correct and flawless as the above answer is, it doesn't give you (who are not familiar with differential geometry) much insight. Both cylinder and sphere look curved after all and it is hard to spot the Riemann tensor at first sight. Next comes Behzad's ingenious answer: "A surface can be made with paper only if it is bent in one direction and is straight in the other directions. As a cylinder is curved only around it's axis and straight along it." Now this is of higher practical value since you can check this condition very quickly just by watching the surface. This note provides a proof of the statement and somehow generalizes it.

Let $\mathcal{M}$ be a p dimensional surface embedded in an $n > p$ dimensional euclidean flat space. Behzad's conjecture reads

"$\mathcal{M}$ is locally flat (and can be made with a p-paper) only if it is bent in one direction at most, and is straight in the other directions."

This general statement does not hold as the following simple example shows. Consider the below 2 dimensional flat paper embedded (bent) in $\mathbb{E}^6$

$$x^1 = \sin(\theta)\sin(\phi)$$

$$x^2 = \sin(\theta)\cos(\phi)$$

$$x^3 = \cos(\theta)$$

$$x^4 = \cos(\theta)\sin(\phi)$$

$$x^5 = \cos(\theta)\cos(\phi)$$

$$x^6 = \sin(\theta)$$

It is easy to check that the induced metric on the surface is

$$ds^2 = 2d\theta^2 + d\phi^2$$

which is a flat 2d surface. However looking at the first 3 x coordinates it resembles a sphere and is therefore bent in more than one direction.

Nevertheless the statement for $p = n - 1$ holds and we will prove it below.

1 The proof

For each point $P \in \mathcal{M}$ define the tangent coordinate system on $\mathbb{E}^n$ to be the Cartesian coordinate system x^i $(i = 1, 2, \cdots, n)$ such that

- P is at $x^i = 0$.
- The first $n - 1$ directions are tangent to the surface. Therefore we can use x^μ with $\mu = 1, 2, \cdots, p$ as a suitable coordinate system near the point P.
- The surface has to be determined by a constraint. Without loss of generality we assume the constraint to be of the form

$$z := x^n(x^\mu) = A_{\alpha\beta} x^\alpha x^\beta + \mathcal{O}(x^3), \qquad A_{\alpha\beta} = A_{\beta\alpha}$$

To compute the Riemann tensor we need to compute the metric tensor to second order in x.

$$\begin{aligned} ds^2 &= dz^2 + \sum_\mu \left(dx^\mu\right)^2 \\ &= 4A_{\alpha\mu}A_{\beta\nu}x^\alpha x^\beta dx^\mu dx^\nu + \sum_\mu \left(dx^\mu\right)^2 + \mathcal{O}(x^3) \end{aligned}$$

In matrix notation

$$G = \mathbf{1} + 4A\vec{x}\vec{x}^T A + \mathcal{O}(x^3)$$

In general, Riemann is

$$R^\mu{}_{\alpha\beta\gamma} = -\partial_\gamma \Gamma^\mu{}_{\alpha\beta} + \partial_\beta \Gamma^\mu{}_{\alpha\gamma} - \Gamma^\mu{}_{\nu\gamma}\Gamma^\nu{}_{\beta\alpha} + \Gamma^\mu{}_{\alpha\gamma}\Gamma^\nu{}_{\gamma\beta}$$

With

$$\Gamma_{\mu\alpha\beta} = \frac{1}{2}\left(\partial_\alpha g_{\beta\mu} + \partial_\beta g_{\alpha\mu} - \partial_\mu g_{\alpha\beta}\right)$$

Turning the crank yields

$$R_{\mu\alpha\beta\gamma} \propto A_{\alpha\beta}A_{\mu\gamma} - A_{\alpha\gamma}A_{\mu\beta}$$

The condition $R_{\mu\alpha\beta\gamma} = 0$ is only nontrivial for two pairs of non repetitive indices $\{\alpha, \mu\}$, $\{\beta, \gamma\}$.

It may be re-written as

$$A_{\alpha\beta} = A_{\alpha\gamma}\frac{A_{\mu\beta}}{A_{\mu\gamma}}$$

Now the left hand side is independent of μ and it can only hold if

$$A_{\alpha\beta} = a_\alpha b_\beta$$

while symmetry insists $a = b$ we get

$$z = \left(a_\mu x^\mu\right)^2 + \mathcal{O}(x^3)$$

Which may be rephrased as

"A paper made surface (a flat manifold embedded in a flat space) of co-dimensionality 1, is curved in not more than one directions."

The converse also holds

"Any patch on a surface of co-dimensionality 1, that is no where bent in more than one directions, can be made with a paper of suitable dimension."

As a corollary, note that supercylinders are not flat and therefore can NOT be made using papers.

2 A Converse Problem and a Physicist's Proof

In this section we consider the converse problem, given a flat p-paper and a 1-form field a_μ defined over it, what are the conditions (if any) on a that guarantee an embedding in $\mathbb{E}^{p+1}$

$$\vec{r} = \vec{r}(x^\alpha)$$

satisfying the below equations?

$$\frac{\partial \vec{r}}{\partial x^\alpha} \cdot \frac{\partial \vec{r}}{\partial x^\beta} = \delta_{\alpha\beta}$$

$$\frac{\partial^2 \vec{r}}{\partial x^\alpha \partial x^\beta} = -\frac{1}{2} a_\alpha a_\beta \hat{n}$$

here $\hat{n}$ denotes the normal to the vectors $\hat{e}_\alpha \equiv \frac{\partial \vec{r}}{\partial x^\alpha}$.