Gravity Notebook

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1 Linearized Gravity

In this first section, we review the linearized gravity theory describing the weak-field limit of the General theory of relativity. By a weak-field limit we mean a small deviation in the spacetime metric from the flat Minkowski metric $\eta_{\mu\nu}$.

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \Leftrightarrow g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$$

with

$$h^{\mu\nu} \equiv \eta^{\mu\alpha}\eta^{\nu\beta}h_{\alpha\beta}$$

The perturbation field $h_{\alpha\beta}$ has a gauge degree of freedom corresponding to infinitesimal diffeomorphisms

$$x^\mu \to x^\mu + \xi^\mu$$

This changes the metric as

$$g_{\mu\nu} \to g_{\mu\nu} - \partial_{\{\mu}\xi_{\nu\}}$$

Now let us define

$$\gamma^{\mu\nu} \equiv h^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}h$$

with $h \equiv h^{\mu\nu}\eta_{\mu\nu}$ being the trace of the perturbation. Under a gauge transformation it is easy to see that

$$\partial_\mu\gamma^{\mu\nu} \to \partial_\mu\gamma^{\mu\nu} - \Box\xi^\nu$$

which may be set to zero via adjusting ξ^α. From now on, we will be working in this gauge. Now let us do the Ricci tensor. In first order approximation, the $\Gamma\Gamma$ terms disappear and this becomes

$$\mathcal{R}_{\alpha\beta} = \frac{1}{2}\eta^{\mu\nu}\big(\partial_\alpha\partial_\mu h_{\beta\nu} + \partial_\beta\partial_\mu h_{\alpha\nu} - \partial_\nu\partial_\mu h_{\beta\alpha} - \partial_\alpha\partial_\beta h_{\mu\nu}\big)$$

Using this and the gauge equation, Einstein's tensor will become

$$\mathcal{G}_{\alpha\beta} = -\frac{1}{2}\Box\gamma_{\alpha\beta} = 2T_{\alpha\beta}$$

and the field equations

$$\Box\gamma_{\alpha\beta} + 4T_{\alpha\beta} = 0$$

In the Newtonian limit, this is

$$\nabla^2\gamma^{00} = -4\rho$$

in comparison with the Newton's law of gravitation, we get

$$\gamma^{\alpha\beta} = \begin{pmatrix} -4\phi_N & & \\ & & \\ & & \end{pmatrix}$$

or equivalently

$$h_{\alpha\beta} = -2\phi_N \begin{pmatrix} 1 & & \\ & 1 & \\ & & \ddots \end{pmatrix}$$

1.1 The energy-momentum pseudo tensor

In the previous part, we developed the linearized theory of gravitation. This may be made exact using the higher order terms. Any solution to the EFE, $G_{\alpha\beta}[g_{\mu\nu}] = 2T_{\mu\nu}$, automatically satisfies $\nabla_\mu T^{\mu\nu} = 0$. However, when using the perturbation theory, one would like to think of the space-time as flat and write the energy-momentum conservation law as $\partial_\mu \hat{T}^{\mu\nu} = 0$ with $\hat{T}^{\mu\nu} = T^{\mu\nu} + t^{\mu\nu}$ being the modified momentum energy tensor. From this perspective, the pseudo-tensor[1] $t^{\mu\nu}$ corresponds to the gravitational field. Now our task would be to show that it is really possible to find such a pseudo tensor. Let us begin by writing a form of the Bianchi identity

$$\nabla_\mu G^{\mu\nu} = 0$$

This may be written in a perturbation series form

$$\big(\nabla^{(0)}_\mu[h] + \nabla^{(1)}_\mu[h] + \cdots\big)\big(G^{(0)\mu\nu}[h] + G^{(1)\mu\nu[h]} + G^{(2)\mu\nu}[h] + \cdots\big) = 0$$

Where $\nabla^{(0)}_\mu = \partial_\mu$ and $G^{(0)}_{\alpha\beta} = 0$. Since this needs to be true for all orders of perturbation, one gets infinitely many simultaneous equations

$$\partial_\mu G^{(1)\mu\nu} = 0$$
$$\partial_\mu G^{(2)\mu\nu} + \nabla^{(1)}_\mu G^{(1)\mu\nu} = 0$$
$$\vdots$$

Consistency of the problem, guarantees $\nabla_\mu T^{\mu\nu} = 0$ or

$$\big(\partial_\mu + \nabla^{(1)}_\mu[h] + \cdots\big)T^{\mu\nu} = 0$$

Finally, we use the EFE $G^{(1)\mu\nu}[h] + \cdots = 2T^{\mu\nu}$ to get

$$\partial_\mu T^{\mu\nu} + \nabla^{(1)}_\mu[h]\frac{1}{2}\big(G^{(1)\mu\nu} + \cdots\big) + \cdots = 0$$

This may be truncated as

$$\boxed{\partial_\mu\big(T^{\mu\nu} - \frac{1}{2}G^{(2)\mu\nu}[h]\big) = \mathcal{O}(h^3)}$$

Looking at the truncated equation, it is tempting to define the (weak field) gravitational energy momentum pseudo tensor as

$$t_{\alpha\beta} \equiv -\frac{1}{2}G^{(2)}_{\alpha\beta}[h]$$

In the vacuum, where $T^{\mu\nu} = 0$, this reduces to

$$t_{\alpha\beta} = -\frac{1}{2}\big(\mathcal{R}^{(2)}_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta}\eta^{\mu\nu}\mathcal{R}^{(2)}_{\mu\nu}\big)$$

[1]This transforms like a tensor under Lorentz transformations but not general diffeomorphisms.

1.2 Gravitational Waves

The linearized EFE clearly shows that the metric perturbation h, satisfies a wave equation with $T^{\mu\nu}$ serving as a source. The natural immediate questions are 1) How to detect these gravitational waves? and 2) What is the effect of this gravitational radiation on the source? (radiation reaction)

Since both questions are physical, the answers need to be gauge invariant. As mentioned before it is possible to use the Transverse Traceless (TT) gauge

$$h^\mu_\mu = 0; \quad \partial_\mu h^{\mu\nu} = 0$$

For a plane wave $h_{\mu\nu} = H_{\mu\nu} e^{ik_\alpha x^\alpha}$, it is easy to show that one can further impose $H_{0\mu} = 0$. Without loss of generality, let the wave vector take the form

$$k_\alpha = \omega(-1, 1, 0, 0, \cdots)$$

The gauge then insists on the form

$$H_{\mu\nu} = \begin{pmatrix} 0 & \cdots & \\ \vdots & 0 & \cdots \\ & \vdots & H_{ab} \end{pmatrix}$$

with H_{ab} being a symmetric, traceless tensor. Our next task would be to find the power, carried by such a wave. Inserting h in the formula for $t_{\alpha\beta}$ we get

$$t_{\alpha\beta} = \frac{1}{8} k_\alpha k_\beta \sum_{a,b} H^2_{ab}$$

The inhomogeneous wave equation $\Box\gamma^{\mu\nu} + 4T^{\mu\nu} = 0$ in $n+1$ dimensions may be solved using time Fourier transform to yield

$$\gamma^{\mu\nu}(t, \vec{r}) = -\frac{2}{\pi} \int dt' d\vec{r}' T^{\mu\nu}(t', \vec{r}') \int_{-\infty}^{+\infty} d\omega |\omega|^{n-2} y\big(|\omega(\vec{r} - \vec{r}')|\big) e^{-i\omega(t-t')}$$

Where y satisfies the radial n dimensional Helmholtz' equation

$$y''(x) + \frac{n-1}{x} y'(x) + y(x) = 0$$

To fix the solution, we need to introduce boundary conditions. Comparison with the static limit fixes the small value asymptotic form of the solution

$$y(x \ll 1) \sim -\frac{\Gamma(n/2)}{2(n-2)\pi^{n/2}} x^{2-n}$$

Also, we assume the causality condition: waves fly away toward spatial infinity and no wave is coming from the infinities. In other words

$$y(x \gg 1) \sim a(x) e^{ix}$$

for a slowly varying amplitude $a(x)$. This fixes the solution except for a multiplicative constant

$$y(x) = A x^{1-n/2} H_{n/2-1}(x)$$

Where H denotes the Hankel function

$$H_\alpha(x) = \frac{J_{-\alpha}(x) - e^{-i\alpha\pi} J_\alpha(x)}{i \sin \alpha\pi}$$

Using the small argument approximation for the Bessel function

$$J_\alpha(x \ll 1) = \frac{1}{\Gamma(\alpha+1)}\big(\frac{x}{2}\big)^\alpha$$

we find the proper constant to be

$$A = \frac{-i\pi}{2(2\pi)^{n/2}}$$

Finally we get our answer as

$$\boxed{\gamma^{\mu\nu}(t,\vec{r}) = \frac{i}{(2\pi)^{n/2}}\int dt' d\vec{r}' T^{\mu\nu}(t',\vec{r}')\int_{-\infty}^{+\infty} d\omega\big(\frac{|\vec{r}-\vec{r}'|}{\omega}\big)^{1-n/2} H_{n/2-1}\big(|\omega(\vec{r}-\vec{r}')|\big)e^{-i\omega(t-t')}}$$

2 Spherically Symmetric Solutions

The most general metric with spherical symmetry in $(D+1)+1$ dimensions ($g^\mu_\mu = D+2$) is given by[2]

$$ds^2 = -F(t,r)dt^2 + G(t,r)dr^2 + 2H(t,r)dtdr + r^2 d\Omega^2 \tag{1}$$

With

$$d\Omega^2 = \sigma_{IJ}d\theta^I d\theta^J = d\theta^{12} + \cos^2(\theta^1)\Big(d\theta^{22} + \cos^2(\theta^2)\big(d\theta^{32} + \cos^2(\theta^3)(\cdots)\big)\Big) \tag{2}$$

$$\approx \sum_{I=1}^{D}\Big(1 - \sum_{J=1}^{I-1}(\theta^J)^2\Big)(d\theta^I)^2 \qquad \theta^I \ll 1$$

In matrix form

$$g_{\mu\nu} = \begin{pmatrix} -F & H & 0 \\ H & G & 0 \\ 0 & 0 & r^2\sigma \end{pmatrix}$$

$$g^{\mu\nu} = \begin{pmatrix} \frac{-G}{FG+H^2} & \frac{H}{FG+H^2} & 0 \\ \frac{H}{FG+H^2} & \frac{F}{FG+H^2} & 0 \\ 0 & 0 & \frac{\sigma^{-1}}{r^2} \end{pmatrix}$$

The spherical components of the Christoffel symbols, Γ^I_{JK}, for the g metric may be correctly computed using $d\Omega^2$ alone instead of ds^2. At $\theta^I = 0$ we have

$$\sigma_{IJ} = \delta_{IJ}$$

$$\Gamma^I_{JK} = 0$$

$$\partial_I\Gamma^J_{KL} = \delta_{IJ}\delta_{KL}\mathbf{1}_{[I<K]} - \delta_{IK}\delta_{JL}\mathbf{1}_{[I<J]} - \delta_{IL}\delta_{JK}\mathbf{1}_{[I<J]}$$

This is all we will need to compute the Einstein tensor and write down the Einstein's field equations; higher order derivatives do not matter here. The choice $\theta^I = 0$ also does not lead to a loss of generality since spherical symmetry is assumed. The other non-zero components of the Christoffel symbols for the spherically symmetric metric at $(t, r, 0, \cdots, 0)$ are

$$\Gamma^t_{tt} = \frac{G\partial_t F + H\partial_r F + 2H\partial_t H}{2(H^2+FG)}$$

[2]We will always be assuming that $D > 0$. The special case of a 1+1 dimensional world will be discussed in the last section.

$$\Gamma^t_{tr} = \Gamma^t_{rt} = \frac{G\partial_r F + H\partial_t G}{2(H^2+FG)}$$
$$\Gamma^t_{rr} = \frac{H\partial_r G + H\partial_t G - 2G\partial_r H}{2(H^2+FG)}$$
$$\Gamma^t_{IJ} = \frac{-rH\sigma_{IJ}}{(H^2+FG)}$$
$$\Gamma^r_{tt} = \frac{-H\partial_t F + F\partial_r F + 2F\partial_t H}{2(H^2+FG)}$$
$$\Gamma^r_{tr} = \Gamma^r_{rt} = \frac{-H\partial_r F + F\partial_t G}{2(H^2+FG)}$$
$$\Gamma^r_{rr} = \frac{F\partial_r G - H\partial_t G + 2H\partial_r H}{2(H^2+FG)}$$
$$\Gamma^r_{IJ} = \frac{-rF\sigma_{IJ}}{(H^2+FG)}$$
$$\Gamma^I_{rJ} = \Gamma^I_{Jr} = \frac{\delta_{IJ}}{r}$$

The Ricci tensor

$$\mathcal{R}_{\alpha\beta} = \partial_\mu \Gamma^\mu_{\alpha\beta} - \partial_\alpha \Gamma^\mu_{\beta\mu} + \Gamma^\mu_{\alpha\beta}\Gamma^\nu_{\mu\nu} - \Gamma^\mu_{\alpha\nu}\Gamma^\nu_{\beta\mu}$$

is then found to have the non-zero components

$$\begin{aligned}
\mathcal{R}_{tt} = \frac{1}{2(H^2+FG)^2}\Big\{ & 2FH^2\partial_r\partial_t H + FH^2\partial_r^2 F - FG(\partial_t F)(\partial_r H) + 2F^2G\partial_r\partial_t H + F^2G\partial_r^2 F + \frac{1}{2}FH(\partial_t F)(\partial_r G) \\
& -2FH(\partial_t H)(\partial_r H) - F^2(\partial_r G)(\partial_t H) - FH(\partial_r F)(\partial_r H) - \frac{1}{2}F^2(\partial_r F)(\partial_r G) - FH^2\partial_t^2 G - F^2G\partial_t^2 G \\
& -\frac{1}{2}FH(\partial_r F)(\partial_t G) + \frac{1}{2}F^2(\partial_t G)^2 + \frac{D}{r}(H^2+FG)(F\partial_r F + 2F\partial_t H - H\partial_t F) \\
& -\frac{1}{2}FG(\partial_r F)^2 + FH(\partial_t G)(\partial_t H) + \frac{1}{2}FG(\partial_t F)(\partial_t G)\Big\}
\end{aligned}$$

$$\begin{aligned}
\mathcal{R}_{tr} = \mathcal{R}_{rt} = \frac{1}{2(H^2+FG)^2}\Big\{ & -H^3\partial_r^2 F + H^2(\partial_r H)(\partial_r F) + \frac{1}{2}H^2(\partial_r F)(\partial_t G) - FGH\partial_r^2 F + \frac{1}{2}HF(\partial_r F)(\partial_r G) \\
& +\frac{1}{2}HG(\partial_r F)^2 + H^3\partial_t^2 G - H^2(\partial_t H)(\partial_t G) - \frac{1}{2}H^2(\partial_t F)(\partial_r G) - 2H^3\partial_r\partial_t H + 2H^2(\partial_r H)(\partial_t H) + FGH\partial_t^2 G \\
& -2FGH\partial_r\partial_t H + FH(\partial_r G)(\partial_t H) - \frac{1}{2}FH(\partial_t G)^2 - \frac{1}{2}GH(\partial_t F)(\partial_t G) + GH(\partial_t F)(\partial_r H) \\
& +\frac{D}{r}(H^2+FG)(F\partial_t G - H\partial_r F)\Big\}
\end{aligned}$$

$$\begin{aligned}
\mathcal{R}_{rr} = \frac{1}{2(H^2+FG)^2}\Big\{ & -2GH^2\partial_r\partial_t H + GH^2\partial_t^2 G + FG(\partial_r G)(\partial_t H) - 2FG^2\partial_r\partial_t H + FG^2\partial_t^2 G + 2GH(\partial_t H)(\partial_r H) \\
& -GH(\partial_t G)(\partial_t H) - \frac{1}{2}GH(\partial_t F)(\partial_r G) + G^2(\partial_t F)(\partial_r H) - \frac{1}{2}G^2(\partial_t F)(\partial_t G) - GH^2\partial_r^2 F - FG^2\partial_r^2 F + \frac{1}{2}G^2(\partial_r F)^2 \\
& +\frac{1}{2}GH(\partial_r F)(\partial_t G) - \frac{1}{2}FG(\partial_t G)^2 + \frac{1}{2}FG(\partial_r F)(\partial_r G) + GH(\partial_r F)(\partial_r H) + \frac{D}{r}(H^2+FG)(F\partial_r G - H\partial_t G + 2H\partial_r H)\Big\}
\end{aligned}$$

$$\mathcal{R}_{IJ} = \delta_{IJ}\Big\{(D-1)\Big(1 - \frac{F}{H^2+FG}\Big) - \frac{r}{H^2+FG}\Big[\partial_t H + \partial_r F - \frac{1}{2(H^2+FG)}(H\partial_t + F\partial_r)(H^2+FG)\Big]\Big\}$$

2.1 Static Space-times

A static, spherically symmetric space-time is identified by $H = 0$ and $\partial_t F = \partial_t G = 0$. The constituent matter will be composed of perfect fluids and electromagnetic fields, although this is *not* the most general budget.

2.1.1 Electrostatics on a spherically symmetric space-time

The Electromagnetic fields form a two form $F_{\alpha\beta}$ satisfying $dF = 0$ and $\nabla_\mu F^{\mu\nu} = J^\nu$. The first equation proves the existence of a vector potential A_μ such that $dA = F$. Clearly, the theory is gauge invariant under the gauge transformation $A \to A + d\Lambda$; we would usually work in the Lorenz gauge in which $\nabla_\mu A^\mu = 0$.

The electromagnetic stress-energy tensor may be derived from the Lagrangian

$$\mathcal{L}_{EM} \equiv J^\mu A_\mu - \frac{1}{4} g^{\alpha\beta} g^{\mu\nu} F_{\alpha\mu} F_{\beta\nu}$$

via the Hilbert recipe

$$T_{\mu\nu} \equiv -2\frac{\partial \mathcal{L}}{\partial g^{\mu\nu}} + g_{\mu\nu}\mathcal{L} = F_{\alpha\mu} F^\alpha_{\ \nu} + g_{\mu\nu}\Big(J^\alpha A_\alpha - \frac{1}{4} F^{\alpha\beta} F_{\alpha\beta}\Big)$$

At this point we are ready to analyze the electrostatics of a spherically symmetric space-time. The most general four potential is

$$A = \phi(r) dt$$

which yields

$$F = \phi'(r) dr \wedge dt$$

The maxwell equations ($\nabla_\mu F^{\mu\nu} = \rho_e V^\nu$) with $V^\mu = (1/\sqrt{F}, 0, 0, 0)$ read

$$\nabla_\mu F^{\mu\nu} = \frac{1}{\sqrt{-g}} \partial_\mu \sqrt{-g} F^{\mu\nu} = \rho_e V^\nu$$

$$\Rightarrow \boxed{\frac{d}{dr}\Big(\frac{\phi' r^D}{\sqrt{FG}}\Big) + \sqrt{G} r^D \rho_e = 0}$$

2.1.2 Einstein-Maxwell Equations

In order to draw a connection between the metric components and the matter content, we must use Einstein's field equations, in this case

$$\mathcal{R}_{\mu\nu} = 2\big\{T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T\big\}$$

Adding a perfect fluid in the last minute, these read

$$tt: \quad \frac{F''}{2G} - \frac{F'G'}{4G^2} + \frac{DF'}{2rG} - \frac{F'^2}{4FG} = \frac{D}{2}\frac{\phi'^2}{G} + D\rho_e \phi \sqrt{F} + (D+1)pF + \rho F$$

$$rr: \quad -\frac{F''}{2F} + \frac{F'^2}{4F^2} + \frac{F'G'}{4FG} + \frac{DG'}{2rG} = -\frac{D\phi'^2}{2F} - \frac{D\rho_e \phi G}{\sqrt{F}} - (D-1)pG + \rho G$$

$$IJ: \quad (D-1)(1 - \frac{1}{G}) - \frac{r}{2G}\Big(\frac{F'}{F} - \frac{G'}{G}\Big) = r^2\Big[\frac{4-D}{2}\frac{\phi'^2}{FG} - \frac{D\rho_e\phi}{\sqrt{F}} - (D-1)p + \rho\Big]$$

Adding the first two equations yield

$$\boxed{F = \frac{1}{G} \exp\Big\{ -\frac{4}{D} \int_r^\infty r' G(p+\rho) \mathrm{d}r' \Big\}}$$

Then the IJ equation yields

$$\boxed{\frac{d}{dr}\Big[r^{D-1}(1-\frac{1}{G})\Big] = r^D\Big[\frac{4-D}{2}\frac{\phi'^2}{FG} - \frac{D\rho_e\phi}{\sqrt{F}} + (\frac{2}{D} - D + 1)p + (1+\frac{2}{D})\rho\Big]}$$

2.2 Schwarzschild Solutions

A Schwarzschild solution is a static spherically symmetric space-time ($H = 0,\ \partial_t g_{\mu\nu} = 0$[3]) which satisfies $T_{\alpha\beta} = 0\ \forall r > 0$. For $D > 0$ this immediately implies $\mathcal{R}_{\alpha\beta} = 0$

2.2.1 The $D = 1$ Case

For a $2+1$ dimensional space-time, starting from $\mathcal{R}_{IJ} = 0$ and using a time deceleration or acceleration one may redefine time in a way that $F = 1$ is satisfied. This in turn yields

$$G = const.$$

Or

$$ds^2 = -dt^2 + Gdr^2 + r^2 d\phi^2$$

Any non-trivial ($G \neq 1$) solution is singular at the origin $r = 0$. Such singularities are called *conic* since they are similar to the *visible* singularity at the tip of a paper cone when $G > 1$. The reader can convince himself using the paper cone picture that parallel transportation of a vector around a loop that contains the origin, leads to a finite rotation of the vector, the angle of which is independent of the shape/size of the loop. This clearly suggests that the Riemann tensor has a delta function singularity at $r = 0$.

2.2.2 The $D > 1$ Case

The equation $\mathcal{R}_{tt} = 0$ may be integrated as

$$\frac{F' r^D}{\sqrt{FG}} = const.$$

substituting G into $\mathcal{R}_{rr} = 0$ and integrating then gives

$$F' r^D = const.$$

comparison with Newtonian gravity[4] and asymptotic flatness then yield the solution

$$ds^2 = -\big(1 - (r_s/r)^{D-1}\big)dt^2 + \frac{dr^2}{1-(r_s/r)^{D-1}} + r^2 d\Omega^2$$

[3] A fancier way to say this, would be to assert that the space time possesses a *timelike* Killing vector that commutes with other Killing vectors related to spherical symmetry.

[4] To see a more detailed discussion of the weak-field linearized gravity, see subsection 2.3 below.

with the Schwarzschild radius, r_s, defined as[5]

$$r_s \equiv \frac{1}{\sqrt{\pi}}\Big[\frac{M\Gamma\big(\frac{D-1}{2}\big)}{2\pi}\Big]^{\frac{1}{D-1}}$$

From now on, we assume $r_s = 1$. These solutions all exhibit a coordinate singularity at the event horizon $r = r_s$ along with a physical singularity at $r = 0$. To see that the $r = 1$ singularity is not a curvature singularity and simply a coordinate singularity, we will shortly define the Kruskal coordinates.

The first task would be to find the paths of outgoing and ingoing light rays. We would define functions u and v such that an outgoing (ingoing) light ray would move on a path $u = const.$ ($v = const.$). It is easy to check that possible definitions include

$$u \equiv t - r^*, \qquad v \equiv t + r^*$$

$$r^* \equiv \int_a^r \frac{dz}{1 - z^{1-D}}, \qquad a \in (1, +\infty)$$

$$ds^2 = -(1 - r^{1-D})dudv + r^2 d\Omega^2$$

$$r = r^{*-1}\big(\frac{v-u}{2}\big)$$

This definition would be a proper well-defined one when one considers only the exterior region $r > 1$. Near $r = 1$, the integral diverges logarithmically

$$\lim_{r\to 1} r^* = -\infty, \qquad \lim_{r\to\infty} r^* = \infty$$

$$\lim_{r\to 1} u = \infty, \qquad \lim_{r\to\infty} u = -\infty$$

$$\lim_{r\to 1} v = -\infty, \qquad \lim_{r\to\infty} v = \infty$$

This means the coordinates (u, v) map the exterior region to the whole $\mathbb{R}^2$ plane; there simply is no room for the interior region to be addressed. To make more room, we need a map that compresses $\mathbb{R}$ (reversibly) to one of its subsets , say $\mathbb{R}^+$. A simple example would be the exponential map.

$$U = -e^{-u/2}, \qquad V = e^{v/2}$$

$$ds^2 = -4e^{-r^*}(1 - r^{1-D})dUdV + r^2 d\Omega^2$$

This makes sense only for $U < 0$, $V > 0$. Our first extension will be to use the $U > 0$ region to address the interior solution $r < 1$. It is possible to show that the definition

$$U = e^{-u/2}$$

would make for a consistent extension, provided that one changes the definition of r^* properly to set $r^*(0) = 0$. This maps the physical singularity $r = 0$ to $UV = 1$. Finally, we extend the V coordinate to the negative values. The last step, yields a manifold that is geodesically complete i.e. no geodesic meets the end of the spacetime in finite affine parameter value. This means that we are done extending the solution. So far our solution looks like

[5] This might look a little different from the most popular conventions. Usually, people work in a system of units which guarantees $G = 1$. We however use the rather different convention $4\pi G = 1$ for the ordinary 4 dimensional spacetime. In general we insist that

$$G_{\alpha\beta} = 2T_{\alpha\beta}$$

holds in any dimension. The two conventions are equivalent to the choice of making the Coulomb's constant, k, unity or the vacuum permeability, ε_0. We use the latter convention. Nevertheless, this should not make you worry since most of the time, we further assume $r_s = 1$. This will clearly lead to results that are left invariant under a change of units since $\frac{\ell}{r_s}$ is dimensionless for any physical length ℓ.

¿¿¿¿????

Using the map

$$(\tilde{U}, \tilde{V}) = (\arctan(U), \arctan(V))$$

the so called Penrose diagram will represent the causal structure of the spacetime in a nice, compactified picture.

¿¿¿¿????

Next, we will focus on the motion of particles in this spacetime. To find the orbital motions for both massive and massless particles in the exterior region, we use the original (t, r, θ^I) Schwarzschild coordinates. The 4-velocity satisfies

$$u^\mu u_\mu = -\kappa$$

$$\kappa = \begin{cases} 0 & \text{Null,} \qquad \text{Massless} \\ 1 & \text{Timelike,} \qquad \text{Massive} \end{cases}$$

Without loss of generality (and using the spherical symmetry) one may assume $\theta^I = 0$ for all $1 \leq I < D$.

$$-\kappa = -(1 - r^{1-D})\big(\frac{dt}{d\tau}\big)^2 + \frac{(dr/d\tau)^2}{1 - r^{1-D}} + r^2\big(\frac{d\phi}{d\tau}\big)^2$$

One may also use the fact that ∂_t and ∂_ϕ are killing fields, to find

$$\mathcal{E} \equiv -(\partial_t)^\mu u_\mu = (1 - r^{1-D})\frac{dt}{d\tau} = const.$$

$$\ell \equiv (\partial_\phi)^\mu u_\mu = r^2 \frac{d\phi}{d\tau} = const.$$

which immediately yield the radial equation of motion

$$\big(\frac{dr}{d\tau}\big)^2 + \big(\frac{\ell^2}{r^2} + \kappa\big)(1 - r^{1-D}) = \mathcal{E}^2$$

This behaves as a particle inside the effctive potential

$$V = \big(\frac{\ell^2}{r^2} + \kappa\big)(1 - r^{1-D})$$

Our first calculation is aimed to find the maximum proper time for an observer before hitting the $r = 0$ singularity when starting from the $r = 1$ event horizon. Clearly, to maximize the proper time we need to minimize $ds^2 < 0$. This immediately suggests $\theta^I = const. \Leftrightarrow \ell = 0$. Inserting this to the equation we just found yields

$$d\tau = \frac{dr}{\sqrt{r^{1-D} - 1}}$$

which in turn may be used to produce the result below.

$$\tau_{max} = \frac{2r_s}{D-1}\int_0^{\frac{\pi}{2}} \cos^{\frac{2}{D-1}}(x)dx \sim \sqrt{\frac{\pi}{2eD}}, \quad \lim_{D\to\infty} \tau_{max} = 0$$

While it is not so difficult to show that in the classical limit, $r \gg 1$, $\tau \approx t$, $\ell^2 \sim r^{3-D}$ the geodesic equation of motion for massive particles reduces to that of a Newtonian model, one may wonder what is the first general relativistic correction to the classical orbital motion. The calculations necessary to answer this question are postponed to the next section. The answer to be found there is that a general closed elliptical orbit,

3. Calculate the effect of the general relativistic correction on the orbit of a massive particle in the usual $D = 2$ case.

Answer: Apsidal Precession

4. Find the angle ψ by which a light ray will be deflected due to the gravitation of a massive star in the limit of large collision parameters compared to the Schwarzschild radius, $b \gg 1$. Show that a distant star would appear as a ring due to this *lensing* effect. **More than the answer:**

$$\psi = -\pi + 2\int_0^{x_0} \frac{dx}{\sqrt{1 - x^2\big(1 - (\frac{x}{b})^{D-1}\big)}}, \qquad 1 - x_0^2\big(1 - (\frac{x_0}{b})^{D-1}\big) = 0^6$$

$$\psi = \frac{D}{b^{D-1}}\int_0^1 dx(1-x^2)^{\frac{D-2}{2}} + \mathcal{O}(b^{2-2D}) = \frac{\sqrt{\pi}D}{2b^{D-1}}\frac{\Gamma\big(\frac{D}{2}\big)}{\Gamma\big(\frac{D+1}{2}\big)} + \mathcal{O}(b^{2-2D})$$

Note that for $b < b_c = \big(\frac{D+1}{2}\big)^{\frac{1}{D-1}}\sqrt{\frac{D+1}{D-1}}$, *Light gets absorbed!*

5. Generalise the concept of scattering cross sections and calculate them for low energy (drop the β^2 terms) massive objects when thrown toward a Schwarzschild black hole.

$$\frac{d\sigma}{d\Omega} = \Big(\frac{b}{\sin\psi}\Big)^{D-1}\Big|\frac{\partial\psi}{\partial b}\Big|^{-1}$$

6. For a massive particle

a) Show that for $D > 2$ all circular orbits are unstable.

b) For any D, show that no circular orbit exists in $r \le \big(\frac{D+1}{2}\big)^{\frac{1}{D-1}}$.

c) For $D = 2$, Show that stable circular orbits satisfy $r > 3$.

For a massless particle

d) Find the effective potential and show that there is an unstable circular orbit for all D with the radius calculated in part (b). Come up with an intuitive explanation.

e) Show that in Newtonian gravity, a circular orbit with velocity β, occurs at a radius $r_c(\beta)$ which is half the radius $r_e(\beta)$ at which the escape velocity reaches β. This means that a circular orbit for light should not occur *outside* the horizon in the Newtonian theory. For all D, calculate the functions $r_c(\beta)$, $r_e(\beta)$ the velocity β^* that satisfies $r_c(\beta^*) = r_e(\beta^*)$. Use the definition $\beta = \sqrt{\frac{dx^i dx_i}{-dx^0 dx_0}}$

Answer:

$$r_c(\beta) = \Big[1 + \frac{D-1}{2\beta^2}\Big]^{\frac{1}{D-1}} \qquad r_e(\beta) = \beta^{\frac{-2}{D-1}} \quad \Rightarrow \quad \beta^* = \sqrt{\frac{3-D}{2}}$$

2.3 Reissner-Nordstrom Solutions

$\nabla_\mu F^{\mu\nu} = 0$, gives

$$\frac{\phi' r^D}{\sqrt{FG}} = const. =: \mathcal{C}$$

6 For a massive particle with asymptotic velocity β, the formula will become

$$\psi = -\pi + 2\beta\int_0^{x_0} \frac{dx}{\sqrt{1 - (1 - \beta^2 + \beta^2 x^2)\big(1 - (x/b)^{D-1}\big)}}$$

Comparison with flat space-time electrostatics, $F = G = 1$ motivates the more natural definition

$$\mathcal{C} = -\frac{Q\Gamma\left(\frac{D+1}{2}\right)}{2\pi^{\frac{D+1}{2}}}$$

The only nonvanishing components of the stress-energy tensor are

$$T_{tt} = \frac{\left(\phi'\right)^2}{2G}$$

$$T_{rr} = \frac{-\left(\phi'\right)^2}{2F}$$

Adding the tt and rr component of EFE we immediately get $FG = const$. The asymptotic flatness puts the constant equal to unity.

$$FG = 1$$

Finally subtracting the two equations gives

$$\frac{2\mathcal{C}^2/D}{r^{2D-1}} = \frac{D-1}{r}(1-F) - F'$$

to solve this, it is convenient to use the substitution $f \equiv 1 - F$

$$\frac{df}{dr} + \frac{D-1}{r}f = \frac{Q^2\Gamma^2\left(\frac{D+1}{2}\right)}{2D\pi^{D+1}}r^{1-2D}$$

2.3.1 The $D = 1$ Case

$$F = \frac{Q^2}{2\pi^2}\log\left(\frac{C}{r}\right)$$

2.3.2 The $D > 1$ Case

$$fr^{D-1} + \frac{Q^2\Gamma^2\left(\frac{D+1}{2}\right)}{2D(D-1)\pi^{D+1}}r^{1-D} = const.$$

Using our Schwarzschild solution, we recognize this constant as r_s^{D-1}. Defining the charge Schwarzschild radius in the same way as before

$$\ell_s \equiv \frac{1}{\sqrt{\pi}}\left[\frac{Q\Gamma\left(\frac{D-1}{2}\right)}{2\pi}\right]^{\frac{1}{D-1}}$$

We get

$$F = 1 - \left(\frac{r_s}{r}\right)^{D-1} + \frac{D-1}{2D}\left(\frac{\ell_s}{r}\right)^{2(D-1)}$$

Note that for

$$\frac{|Q|}{M} > \sqrt{\frac{D}{2(D-1)}}$$

No coordinate singularity exists! This describes a *naked* spacelike singularity. Many physicists believe that such singularities should not exist (Cf. cosmic censorship conjecture). If this is to be true, there should be no way to make a Reissner-Noerdstrom black hole with $|Q| > |Q|_{max}$. Maximal extension of coordinates and drawing the penrose diagram for this case is actually very simple. Like before, define the null coordinates

$$u \equiv t - r^*, \quad v \equiv t + r^*, \quad r^* = \int_0^r \frac{dr}{F(r)}$$

Further extension of coordinates is not necessary; all the geodesic either extend to infinities or end in singularities. The Penrose diagram will look like

¿¿¿¿????

The second case to consider is that of an unsaturated solution; $Q < Q_{max}$. Here, there are two different horizons at two different roots of the metric component $F(r)$.

$$r_\pm = \Big[\frac{2}{\alpha}\big(1 \mp \sqrt{1-\alpha}\big)\Big]^{\frac{-1}{D-1}}, \quad \alpha \equiv \frac{2(D-1)}{D}\big(\frac{Q}{M}\big)^{2(D-1)}$$

Let us begin in the outermost region$\cdots$

To find the geodesic motions, we repeat our previous method to get the equation

$$\big(\frac{dr}{d\tau}\big)^2 + \big(\frac{\ell^2}{r^2} + \kappa\big)F(r) = \mathcal{E}^2$$

with

$$\mathcal{E} \equiv F(r)\frac{dt}{d\tau}, \quad \ell \equiv r^2\frac{d\phi}{dt}$$

Exercises

1. For isotropic electromagnetic radiation, show that $\rho = (D+1)p$ which immediately implies

$$T^\mu_\mu = 0$$

Use this to prove that an isotropic radiation in $D \neq 2$ satisfies

$$F^{\alpha\beta}F_{\alpha\beta} = 0$$

2. Show that no massive (timelike) path can ever reach the $r = 0$ singularity in finite proper time (Assume $Q \neq 0$).

3. Show that in the classical limit, $r \gg 1$, $\tau \approx t$, $\ell^2 \sim r^{3-D}$ the geodesic equation of motion for massive particles reduces to that of a Newtonaian model.

4. Describe the effect of the general relativistic correction on the orbit of a massive particle in the usual $D = 2$ case.

Answer: Apsidal Precession; The orbit precesses $\frac{3\pi}{2\ell^2}(1 - \frac{\alpha^2}{4})$ radians per revolution.

2.4 Dusty Solutions

By a dusty solution, (g, V, ρ), we mean a solution for the metric $g_{\mu\nu}(x)$, a mass density $\rho(x)$ and a velocity field $V^\mu(x)$ that satisfy the EFE for the dust model $p = 0$.

$$\mathcal{R}_{\alpha\beta} - \frac{1}{2}\mathcal{R}g_{\alpha\beta} = 2T_{\alpha\beta} \qquad T_{\alpha\beta} = \rho V_\alpha V_\beta$$

The velocity normalization condition

$$V_\mu V^\mu + 1 = 0$$

And the geodesic equation of motion

$$V^\mu \nabla_\mu V^\nu = 0$$

Proposition 1. Corresponding to any metric g satisfying

$$\text{rank}\big(\mathcal{R}_{\alpha\beta}[g] - \frac{1}{2}\mathcal{R}[g]g_{\alpha\beta}\big) \leq 1$$

exists a dusty solution and vice versa.

Here "rank" means its linear algebraic meaning and should not be confused with the tensor algebraic meaning of rank i.e. number of indices.

Proof: Any matrix with rank less than two may be written as

$$\mathcal{R}_{\alpha\beta} - \frac{1}{2}\mathcal{R}g_{\alpha\beta} = U_\alpha U_\beta$$

Let $\rho \equiv \frac{\mathcal{R}}{2}$ and $V_\mu \equiv \frac{U_\mu}{\sqrt{\mathcal{R}}}$. Now it is easy to see that (g, V, ρ) satisfies all above conditions.[7] The inverse part is trivial to prove.

3 Axisymmetric Solutions

From now on, our solutions are no longer spherically symmetric. They describe rotating bodies, for our discussion to maintain its dimensional generality, we need to discuss how to describe rotations and angular momenta in high dimensional space times.

3.1 On Rotation

Physics as we know it respects a rotational symmetry, meaning the laws of physics are covariant under rotations. In $3+1$ dimensions, this leads to a conserved vector quantity called the angular momentum. It is not hard to show that in general, the quantity

$$J^{ij} \equiv \int dx(x^i T^{j0} - x^j T^{i0})$$

is conserved. This is an antisymmetric tensor in the euclidean space $\mathbb{E}^n$. Therefore it is always possible to block diagonalize it in the form

$$\mathbf{J} = \begin{pmatrix} & -L_1 & & & \\ L_1 & & & & \\ & & & -L_2 & \\ & & L_2 & & \\ & & & & \ddots \end{pmatrix}$$

In such cases, it will be natural to use the spatial coordinate system which consists of the 2 dimensional polar pairs. For even dimensions ($d = 2n$) this is

$$(s^1, \phi^1, s^2, \phi^2, \cdots, s^n, \phi^n)$$

and for odd dimensions ($d = 2n+1$)

$$(s^1, \phi^1, \cdots, s^{2n-1}, \phi^{2n-1}, z)$$

[7] One may argue that V is ill-defined for the case $\mathcal{R}^\mu_\mu = 0$ but note that this will not be a problem since when $\rho = 0$, the V field doesn't really *matter*!

3.2 Axisymmetric Geometry

If a spacetime is going to describe a collapsed rotating object, in its stationary state, it needs to have ∂_t and ∂_{ϕ^i} as Killing fields. It is also guaranteed to have a nonzero time-space metric component; otherwise it would be static. It is clear that changing the direction of the rotation for the rotating object, should not change the preferred direction of any s^i or z coordinate. Therefore the time-space components of metric are entirely azimuthal. It also forces no loss of generality to assume $g_{\phi^i s^j} = 0$, $g_{\phi^i \phi^j} = \delta_{ij}(s^i)^2$ and $g_{s^i z} = g_{\phi^i z} = 0$. Finally we have

$$ds^2 = -A(\underset{\sim}{s}, z)dt^2 + 2\sum_i B_i(\underset{\sim}{s}, z)dtd\phi^i + \sum_i C_i(\underset{\sim}{s}, z)d(s^i)^2 + \sum_i (s^i)^2(d\phi^i)^2 + D(\underset{\sim}{s}, z)dz^2$$

3.3 Kerr Geometry

The 3+1 dimensional rotating black hole solution is the Kerr space time

$$ds^2 = -(1 - \frac{r_S r}{s^2})dt^2 - 2\frac{r_S a r}{s^2}\sin^2\theta\, dt\, d\varphi + \frac{s^2}{\Delta}dr^2 + s^2 d\theta^2 + \frac{\Sigma}{s^2}\sin^2\theta\, d\varphi^2$$

with

$$r_S \equiv \frac{M}{2\pi} \quad ; \quad s^2 \equiv r^2 + a^2\sin^2\theta \quad ; \quad \Delta \equiv r^2 - r_S r + a^2 \quad ; \quad \Sigma = (r^2 + a^2)^2 - a^2\Delta\sin^2\theta$$

One can define several other coordinate systems to make the metric suitable for different purposes. The first to consider is the ingoing Kerr coordinates

$$v \equiv t + r^* \quad ; \quad \psi \equiv \varphi + \chi$$

with

$$r^* \equiv \int^r dr' \, \frac{r'^2 + a^2}{\Delta} \quad ; \quad \chi \equiv \int^r dr' \, \frac{a}{\Delta}$$

The metric becomes

$$ds^2 = -(1 - \frac{r_S r}{s^2})dv^2 + 2dvdr - 2a\sin^2\theta\, drd\psi - 2\frac{r_S a r}{s^2}\sin^2\theta\, dvd\psi + s^2 d\theta^2 + \frac{\Sigma}{s^2}\sin^2\theta\, d\psi^2$$

Then come the Kerr-Schild coordinates

$$X + iY = (r + ia)\sin\theta\, e^{i\psi} \quad ; \quad Z = r\cos\theta \quad ; \quad T = v - r$$

that make the metric into

$$ds^2 = (\eta_{\alpha\beta} + Hl_\alpha l_\beta)dX^\alpha dX^\beta$$

with

$$H \equiv \frac{r_S r}{s^2} \quad ; \quad l^\mu\partial_\mu = \frac{r^2 + a^2}{\Delta}\partial_t - \partial_r + \frac{a}{\Delta}\partial_\varphi$$

The parameter M in this space-time, coincides with the Newtonian, ADM, and Komar mass. The parameter a, is the ratio between the ADM angular momentum and the mass M. A horizon exists only if $a \leq M$. The static observers with

$$V = \frac{\partial_t}{\sqrt{-g_{tt}}}$$

are only physical outside the so called ergosphere

$$r = \frac{r_S}{2}\left[1 + \sqrt{1 - (\frac{2a\sin\theta}{r_S})^2}\right]$$

However, stationary observers or those with

$$V = \frac{\partial_t + \omega\partial_\varphi}{\sqrt{-g_{tt} - 2\omega g_{t\varphi} - \omega^2 g_{\varphi\varphi}}}$$

are still viable inside the ergosphere, so long as they are outside of the event horizon

$$g_{tt}g_{\varphi\varphi} = g_{t\varphi}^2$$

It turns out that this is equivalent to

$$\Delta = 0$$

4 Shape of a Kerr Black Hole

Looking at a black hole system with a telescope from a far distance, what does one see? This is the question that I try to answer to some extent in this section. In the absence of any material outside the black hole, there is no light to be captured by the telescope (neglecting the Hawking radiation of course). The apparent *shape* of the black hole is in fact the result of the presence of hot, luminescent matter in the vicinity of the black hole. Here, I neglect the effect of such stuff on the geometry of space time. I am simply interested in knowing what a bright circular ring looks like from afar. In the absence of a black hole, the circular ring, will look like an oval; however, a black hole will bend the light in a way that distorts the image.

Consider a local (i.e. near origin) set of stationary point sources $\mathcal{S} \subset \mathbb{R}^3$ in a stationary space-time with asymptotically flat geometry. One can follow a null geodesic, emanating from the point source $p \in \mathcal{S}$ all the way to infinity (assuming it does not repeat its trajectory or fall into the black hole) and find its asymptotic direction in terms of its initial direction. In a generic case, one will find that for a given asymptotic direction $\hat{\mathbf{n}}$, there are finitely many initial directions in which the null geodesic can start in order to escape in the direction $\hat{\mathbf{n}}$. Then, solving the geodesic equation, it is possible to find the impact parameter $\mathbf{b}$ for such directions. The impact parameter is a three vector such that

$$\hat{\mathbf{n}}.\mathbf{b} = 0$$

and the asymptotic linear trajectory is given by the parametric line

$$\mathbf{x}(\lambda) = \lambda\hat{\mathbf{n}} + \mathbf{b}$$

In principle, for each direction $\hat{\mathbf{n}}$, each point source $p \in \mathcal{S}$ and each distince direction that ends in $\hat{\mathbf{n}}$, one finds a $\mathbf{b}$ vector. I put all such possible vectors for a fixed direction $\hat{\mathbf{n}}$, in a subset of the 2D perpendicular plane and call it $\mathcal{B}(\hat{\mathbf{n}})$. As we will see, this set is closely related to the final image of the light source set on a telescope located in a far distance along $\hat{\mathbf{n}}$.

In order to see this, let the telescope be located at $r\hat{\mathbf{n}}$ and assume that all light trajectories have already converged to their asymptote before reaching the telescope. Then, the light hits the telescope if

$$\lambda\hat{\mathbf{m}} + \mathbf{b}(\hat{\mathbf{m}}; p) = r\hat{\mathbf{n}}$$

This equation is solved for the light's propagation direction $\hat{\mathbf{m}}$ as

$$\hat{\mathbf{m}} = \hat{\mathbf{n}} - \frac{\mathbf{b}(\hat{\mathbf{n}}, p)}{r} + \mathcal{O}(\frac{1}{r^2})$$

In other words, the angular displacements are directly proportional to the impact parameters. Therefore, the final image, printed on a 2D sheet of paper, will have bright spots corresponding to $\mathcal{B}(\hat{\mathbf{n}})$.

Working in the units where $r_s = 1$, the metric to work with is

$$ds^2 = -\big(1 - \frac{r}{\Sigma}\big)dt^2 - 2\frac{ra\sin^2\theta}{\Sigma}dt d\varphi + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 + \big(r^2 + a^2 + \frac{ra^2\sin^2\theta}{\Sigma}\big)\sin^2\theta d\varphi^2$$

in which

$$\Sigma = r^2 + a^2\cos^2\theta\,; \quad \Delta = r^2 - r + a^2$$

On a null geodesic, in addition to the conserved nullity condition, there are 3 constants of motion. $\partial_t g_{\alpha\beta} = \partial_\varphi g_{\alpha\beta} = 0$ asserts the conservation of energy

$$E \equiv -\big\langle \partial_t, \frac{d\mathcal{P}}{d\lambda}\big\rangle$$

as well as the angular momentum around the z-axis.

$$L_z \equiv \big\langle \partial_\varphi, \frac{d\mathcal{P}}{d\lambda}\big\rangle.$$

Since we are only interested in the spatial path taken by the light ray, we may set $E = 1$ without loss of generality. This uniquely determines the t and φ components of the geodesic vector field via

$$\begin{pmatrix} g_{tt} & g_{t\varphi} \\ g_{t\varphi} & g_{\varphi\varphi} \end{pmatrix} \begin{pmatrix} \frac{dt}{d\lambda} \\ \frac{d\varphi}{d\lambda} \end{pmatrix} = \begin{pmatrix} -1 \\ L_z \end{pmatrix}$$

as

$$\boxed{\frac{dt}{d\lambda} = \frac{g_{\varphi\varphi} + L_z g_{t\varphi}}{\Delta\sin^2\theta}\,; \quad \frac{d\varphi}{d\lambda} = -\frac{L_z g_{tt} + g_{t\varphi}}{\Delta\sin^2\theta}}$$

The last constant of motion is called Carter's constant and is given by

$$C = \Sigma\big\langle l, \frac{d\mathcal{P}}{d\lambda}\big\rangle\big\langle n, \frac{d\mathcal{P}}{d\lambda}\big\rangle + r^2\big\langle \frac{d\mathcal{P}}{d\lambda}, \frac{d\mathcal{P}}{d\lambda}\big\rangle$$

with

$$l^\mu = \Big(\frac{r^2 + a^2}{\Delta}, 1, 0, \frac{a}{\Delta}\Big)\,; \;\; n^\mu = \Big(\frac{r^2 + a^2}{\Sigma}, -\frac{\Delta}{\Sigma}, 0, \frac{a}{\Sigma}\Big)$$

For the null geodesics that we are considering, this is

$$C = \frac{1}{\Delta}\Big[(r^2 + a^2 - aL_z)^2 - \Sigma^2 V^{r2}\Big]$$

Which determines the r component up to an overall sign as

$$\boxed{\frac{dr}{d\lambda} = \frac{\pm 1}{\Sigma}\sqrt{(r^2 + a^2 - aL_z)^2 - C\Delta}}$$

Finally, the nullity condition reads

$$\frac{g_{\varphi\varphi} + 2L_z g_{t\varphi} + L_z^2 g_{tt}}{\Delta \sin^2\theta} + C = \frac{r^2 + a^2 - aL_z}{\Delta} + \Sigma V^{\theta 2}.$$

Finally, this determines the θ component up to an overall sign

$$\boxed{\frac{d\theta}{d\lambda} = \pm\sqrt{\frac{C\Delta\sin^2\theta + (a^2 r/\Sigma)\sin^4\theta + aL_z\sin^2\theta(1 - 2r/\Sigma) - L_z^2(1 - r/\Sigma)}{\Sigma\Delta\sin^2\theta}}}$$

4.1 Constants of motion in terms of the impact parameters

We are looking for the null geodesic starting from the spatial point $(\ell, \pi/2, \psi)$ and escaping to infinity in the direction $(\theta, \varphi) = (\alpha, 0)$ to meet the telescope. Fixing the time origin, we assume that asymptotically, the geodesic approaches the line equation parametrized as

$$(t, r, \theta, \varphi) = \Big(\lambda,\ \sqrt{\lambda^2 + b_1^2 + b_2^2},\ \arccos\frac{\lambda\cos\alpha + b_2\sin\alpha}{\sqrt{\lambda^2 + b_1^2 + b_2^2}},\ \arctan\frac{b_1}{\lambda\sin\alpha - b_2\cos\alpha}\Big)$$

Using this asymptotic form, we may find the constants of motion. First, there is the energy

$$E \equiv -\big\langle\frac{d\mathcal{P}}{d\lambda}, \partial_t\big\rangle = -g_{tt}\frac{dt}{d\lambda} - g_{t\varphi}\frac{d\varphi}{d\lambda}$$

Equating this with the energy of the asymptotic trajectory, we find

$$\boxed{E = 1}$$

which is consistent with our convention from the last section. The next constant of motion, is the z-component of the angular momentum

$$L_z \equiv \big\langle\frac{d\mathcal{P}}{d\lambda}, \partial_\varphi\big\rangle = g_{\varphi\varphi}\frac{d\varphi}{d\lambda} + g_{t\varphi}\frac{dt}{d\lambda}$$

Once again, comparison with the asymptote yields

$$\boxed{L_z = -b_1\sin\alpha}$$

Computing Carter constant, is a more elaborate work. First, we compute the 4-velocity norm as

$$\big\langle\frac{d\mathcal{P}}{d\lambda}, \frac{d\mathcal{P}}{d\lambda}\big\rangle = \frac{2}{r} + \frac{1 - a^2\sin^2\alpha}{r^2} + \mathcal{O}(\frac{1}{r^3})$$

Then there are the dot products

$$\langle l, \frac{d\mathcal{P}}{d\lambda}\rangle = \frac{1}{r} + \frac{1}{r^2}\big(1 - a^2\sin^2\alpha - ab_1\sin\alpha - \frac{b_1^2+b_2^2}{2}\big) + \mathcal{O}(\frac{1}{r^3})$$

$$\langle l, \frac{d\mathcal{P}}{d\lambda}\rangle = -2 + \frac{1}{r} + \mathcal{O}(\frac{1}{r^2})$$

A careful calculation then yields

$$\boxed{C = (a\sin\alpha + b_1)^2 + b_2^2}$$

4.2 Special Case of a Schwarzschild Black Hole

For a Schwarzschild space-time

$$ds^2 = -(1-1/r)dt^2 + \frac{dr^2}{1-1/r} + r^2 d\Omega^2$$

The equations of motion simplify as

$$\frac{dt}{d\lambda} = \frac{1}{1-1/r} \quad ; \quad \frac{d\varphi}{d\lambda} = \frac{L_z}{r^2\sin^2\theta}$$

$$\frac{dr}{d\lambda} = \pm\sqrt{1 - \frac{C}{r^2}(1-1/r)} \quad ; \quad \frac{d\theta}{d\lambda} = \pm\sqrt{\frac{C}{r^2} - \frac{L_z^2}{r^4\sin^2\theta}}$$

5 Motion of Charged Particles

6 Maximally Symmetric Cosmological Solutions

By a maximally symmetric or cosmological solution, we mean one in which the space is maximally symmetric. Such a solution will admit a metric in the form

$$ds^2 = -dt^2 + g_{ij}dx^i dx^j$$

Where the spatial manifold $d\ell^2 = g_{ij}dx^i dx^j$ is maximally symmetric and therefore satisfies

$$R_{ijkl} = \frac{K}{a^2}(g_{ik}g_{jl} - g_{il}g_{jk})$$

With $a \in \mathbb{R}^+$ and $K \in \{0, \pm 1\}$. In $n+1$ dimensions, this means[8]

$$\mathcal{R}_{ij} = \frac{(n-1)K}{a^2} g_{ij}$$

Maximal symmetry implies spherical symmetry. We pick the polar coordinates

$$d\ell^2 = G(r)dr^2 + r^2 d\Omega^2$$

[8] When we find the solutions, the reader may check that they are manifestly maximally symmetric and satisfy the original condition in terms of the Riemann tensor too.

Using our previous results, the maximal symmetry equations yield non-trivial results only for components rr and IJ. These independently result in the solution

$$G(r) = \frac{1}{1 - \frac{Kr^2}{a^2}}$$

$K = 0$ corresponds to flat space $\mathbb{E}^n$, while $K = +1$, $K = -1$ correspond to $\mathbb{S}^n$ and $\mathbb{H}^n$ respectively. With a little renaming of coordinates, our space-time metric becomes

$$\boxed{ds^2 = -dt^2 + a^2(t)\big[\frac{dr^2}{1 - Kr^2} + r^2 d\Omega^2\big]}$$

It is evident that the temporal coordinate t here, corresponds to the (common) proper time of co-moving observers.

Exercise: Show that free particles in this space-time travel over straight, spatial geodesics while accelerating/decelerating according to

$$E^2 - m^2 \propto a^{-2}(t)$$

Our next goal would be to find the dynamics of such a space-time filled with different types of matter. In that regard let us begin by calculating the Christoffel symbols. To do so, it is best to use the *Cartesian* coordinates

$$ds^2 = -dt^2 + a^2(t)\tilde{g}_{ij}dx^i dx^j$$

with

$$\tilde{g}_{ij} \equiv [\delta_{ij} + \frac{Kx^i x^j}{1 - Kr^2}]$$

In this coordinate system, the Christoffel symbols become

$$\Gamma^t_{ij} = a\dot{a}\tilde{g}_{ij}$$

$$\Gamma^i_{tj} = \Gamma^i_{jt} = \frac{\dot{a}}{a}\delta^i_j$$

$$\Gamma^i_{jk} = Kx^i\tilde{g}_{jk}$$

Homogeneity allows us to calculate the Ricci tensor at $x^i = 0$ and generalize the formula to any point in the space.

$$\mathcal{R}_{tt} = -n\frac{\ddot{a}}{a}$$

$$\mathcal{R}_{ij} = \big[a\ddot{a} + (n-1)(K + \dot{a}^2)\big]\tilde{g}_{ij}$$

Consequently the Einstein tensor is

$$\mathcal{G}_{tt} = \frac{n(n-1)(K + \dot{a}^2)}{2a^2}$$

$$\mathcal{G}_{ij} = -(n-1)\big[\frac{\ddot{a}}{a} + \frac{(n-2)(K + \dot{a}^2)}{2a^2}\big]g_{ij}$$

Before equating this with the matter content, let us know what sort of matter may exist in a maximally symmetric universe. The Energy-Momentum tensor may be written as

$$T = \sum_{\mu} \partial_\mu \otimes P_{(\mu)}$$

Due to symmetry (and sitting at $x^i = 0$) we have

$$P_{(t)} \propto \partial_t; \quad P_{(i)} \propto \partial_t$$

Also, the constant of proportionality in spatial components is the same for all directions. This only has 2 degrees of freedom and it is easy to see that such a tensor may be written as

$$T = pg + (p+\rho)\partial_t \otimes \partial_t$$

Finally, we find the equations of motion for the universe

$$\boxed{\frac{n(n-1)(K+\dot{a}^2)}{4a^2} = \rho}$$

$$-\frac{n-1}{2}\Big[\frac{\ddot{a}}{a} + \frac{(n-2)(K+\dot{a}^2)}{2a^2}\Big] = p$$

It is possible to re-write the second equation (using the time derivative of the first equation) as the conservation law

$$\nabla_\mu T^{\mu 0} = 0 \Rightarrow \boxed{\dot{\rho} + n\frac{\dot{a}}{a}(\rho+p) = 0}$$

The first equation says that space is finite ($\mathbb{S}^n$) only if

$$\rho > \rho_c \equiv \frac{n(n-1)}{4}\frac{\dot{a}^2}{a^2}$$

The matter content may be a mixture of different materials with different equations of state $p = w\rho$. Such an equation of state leads to a conservation law [9]

$$\rho \propto a^{-n(1+w)}$$

Writing ρ as a sum over different constituents

$$\rho = \int dw \frac{\mathcal{A}(w)}{a^{n(1+w)}}$$

is equivalent to

$$\rho = \frac{n(n-1)H_0^2}{4}\int dw \Big(\frac{a_0}{a}\Big)^{n(1+w)}\Big(\tilde{\Omega}(w) - \frac{K}{\dot{a}_0^2}\delta(w+1-2/n)\Big)$$

with

$$a_0 \equiv a(t_0); \quad H_0 \equiv \frac{\dot{a}(t_0)}{a(t_0)}$$

for some arbitrary time t_0, usually referred to as *the present time.* Since the dynamics of the density is taken care of via the a dependence, we find that $\tilde{\Omega}(w) = cte.$. While this choice of constants may seem absurd at first sight, writing the first dynamical equation (the only one which still gives new information) reveals its benefits

$$\int dw \Big(\tilde{\Omega}(w) - \frac{K}{\dot{a}_0^2}\delta(w+1-2/n)\Big) = 1$$

[9] It may look like we are assuming that different types of matter do not interact with each other, however as the next equation suggests, it is (almost) always possible to ascribe amplitudes to each type's density via a Laplace-like transformation.

Finally, calling the expression between parantheses, $\Omega(w)$, we get to summarize the dynamics of our universe as

$$\boxed{\dot{a}^2 = H_0^2 a^2 \int dw \Omega(w) \Big(\frac{a_0}{a}\Big)^{n(1+w)}}$$

for some normalized (but not necessarily positive) density function $\Omega(w)$. This also allows us to find the age, expansion rate and the decelration parameters corresponding to some scaling parameter $a(t) = a_0/(1+z)$ as

$$t(z) = \frac{1}{H_0} \int_0^{(1+z)^{-1}} \frac{dx}{\sqrt{\int dw \Omega(w) x^{2-n(1+w)}}}$$

$$H(z) = H_0 \sqrt{\int dw \Omega(w) (1+z)^{n(w+1)}}$$

For the deceleration parameter we have

$$q = -\frac{a\ddot{a}}{\dot{a}^2} = \frac{n}{2}(\langle w \rangle + 1) - 1$$

Here, the $\langle . \rangle$ denotes an averaging using the density $\Omega(w)(1+z)^{n(w+1)}$

Since a is a positive it is natural to work with its logarithm. In that case, it is preferrable to define

$$b \equiv \log \frac{a}{a_0}; \quad \omega(s) \equiv \frac{\Omega(s/n - 1)}{n}$$

Then the dynamics is

$$\dot{b}^2 = H_0^2 \int ds \omega(s) e^{-sb}$$

Important types of material include Dust ($s = n$), Radiation ($s = n+1$), Curvature[10] ($s = 2$), and Dark energy ($s = 0$). The behavior of a universe filled with only these materials is the same as a classical particle in a potential

$$V(x) \equiv -\big[\omega_D e^{-nx} + \omega_R e^{-(n+1)x} + \omega_C e^{-2x}\big]$$

with energy $E = \omega_\Lambda = 1 - \omega_D - \omega_R - \omega_C$. For $\omega_c \geq 0$ (corresponding to $\mathbb{E}^n$ and $\mathbb{H}^n$) the potential has no extremae. The faith of the universe depends on the energy level ω_Λ. A negative ω_Λ describes a universe starting with a Big Bang and ending in a Big Crunch while a positive ω_Λ describes an ever expanding universe beginning from a Big Bang or its time reversed version. For $\omega_C < 0$ (i.e. $\mathbb{S}^n$) and $n \neq 2$, the potential has a bump of some height V^* and the faith of the universe depends on whether $\omega_\Lambda \leq V^*$ or not. For $\mathbb{S}^2$ universes, the same may be said with the exception that in this case, a positive ω_D could compensate for positive curvature and it is the sign of $\omega_C + \omega_D$ that must be negative in order for a bump to exist.

7 Adding inhomogeneity to the background

Let us look at solutions of the form

$$ds^2 = -e^{2\phi} dt^2 + a^2(t) e^{-2\phi} dx^i dx^j \Big(\delta_{ij} + \frac{K x^i x^j}{1 - K|x|^2}\Big)$$

[10] Remeber how we defined ω to include K as a density.

7.1 The Flat Space ($K = 0$)

The energy momentum tensor is

$$T_{tt} = \frac{1}{2a^2}\Big[3\dot{a}^2 + 3a^2\dot{\phi}^2 - 6a\dot{a}\dot{\phi} + e^{4\phi}\big(2\nabla^2\phi - |\boldsymbol{\nabla}\phi|^2\big)\Big]$$

$$T_{ti} = \frac{\dot{a}}{a}\partial_i\phi - \dot{\phi}\partial_i\phi + \partial_i\dot{\phi}$$

$$T_{ij} = -(\partial_i\phi)(\partial_j\phi) + \frac{1}{2}e^{-4\phi}\Big[- \dot{a}^2 - 2a\ddot{a} + e^{4\phi}|\boldsymbol{\nabla}\phi|^2 + 8a\dot{a}\dot{\phi} + a^2\big(2\ddot{\phi} - 5\dot{\phi}^2\big)\Big]\delta_{ij}$$

This spacetime will not be physically relevant unless for all timelike 4 velocities, we have

$$T_{\mu\alpha}T_{\nu\beta}g^{\mu\nu}V^\alpha V^\beta \leq 0$$

8 Poisson Cosmoi

9 Stability Notions

10 The 1+1 dimensional spacetime

It is possible to show that any 2-dimensional manifold is conformally flat. Assuming the Lorentzian signature, this means

$$ds^2 = \Omega^2(t,x)[-dt^2 + dx^2]$$

It is a simple task to show

$$\mathcal{R}_{\alpha\beta} = \begin{pmatrix} \Box\omega & \\ & -\Box\omega \end{pmatrix}, \qquad \Box \equiv -\partial_t^2 + \partial_x^2, \ \ \omega \equiv \log\Omega$$

This form leads to $T_{\alpha\beta} = 0$; matter and GR can not coexist in 1+1 dimensions.