Rigid Body Motion and Relativity

Koorosh Sadri

Department of Physics, Sharif University of Technology

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Abstract

Starting from the classical mechanics, the rigid motion is generalized to relativistic mechanics. The popular notion of Born rigidity is disputed as a physically consistent candidate. Alternative approaches from fluid mechanics and elastic theory of solids are introduced; however, non of them may be regarded as the final answer to the question of what, in relativstic physics, is a rigid body.

Contents

1 Introduction

The classical rigid body motion contradicts the relativistic theory in two ways. First, for sufficiently large rotating rigid objects, the particular velocity of particles exceeds the speed of light which is strictly forbidden in the relativistic theory; we shall refer to such problems as *kinematic* problems. Another well known example of a kinematic problem is the one known by Ehrenfest's paradox. Consider a rotating gramophone disk. To the owner of the device, at each moment in time, the disk appears as a flat circular disk of radius R and, not surprisingly, of circumference $2\pi R$. However, to an ant sitting on the very edge of the disk the situation looks different. It will observe its distance to the center of rotation to be R again since there are no transverse Lorentz contractions. However, the circumference now seems larger in its rest frame. Therefore the ant measures the circumference to diameter ratio for the *circular* disk to be $2\gamma\pi$. This apparent paradox is alleviated when one notices that the geometry on a rotating disk is not necessarily Euclidean. [1]

There is also a second way in which the rigidity notion contradicts relativity. A long, rigid rod may be used to transmit superluminal signals since moving one end would instantaneously *cause* a corresponding move in the other end. We will be calling this, the *causality* problem.

To describe the motion of an extended body, a convenient way is to determine a velocity vector field at all relevant times and spatial points. The motion of the constituent particles of the extended body are then assumed to follow the integral curves of the given vector field. The body is then called rigid if the velocity field satisfies certain rigidity conditions. This will be our point of view in most of what followswith the exception of the last section where a different way of describing the motion will be more convenient.

2 Rigidity in Newtonian Mechanics

A classical rigid body satisfies the global constraint that the distance between any two particles does not change during the motion. Mathematically, this is equivalent to the following constraint on the velocity field $\mathbf{v}(\mathbf{r}, t)$

$$\big[\mathbf{v}(\mathbf{r}, t) - \mathbf{v}(\mathbf{r}', t)\big].(\mathbf{r} - \mathbf{r}') = 0 \quad \forall \mathbf{r}, \mathbf{r}'$$

For $\mathbf{r}' = \mathbf{r} + \delta\mathbf{r}$ and up to first order in $\delta\mathbf{r}$ this implies

$$\partial_i v_j + \partial_j v_i = 0 \tag{1}$$

Using the symmetry of the second partial derivatives we may write

$$\partial_i\partial_j v_k = -\partial_i\partial_k v_j = -\partial_k\partial_i v_j = \partial_k\partial_j v_i = \partial_j\partial_k v_i = -\partial_j\partial_i v_k = -\partial_i\partial_j v_k$$

which in turn implies

$$\partial_i\partial_j v_k = 0$$

The solution to this equation is

$$\mathbf{v}(\mathbf{r}, t) = \mathbf{v}_0(t) + \Omega(t)\mathbf{r}$$

for some antisymmetric tensor $\Omega(t)$. Such a field also satisfies the *global* constraint and therefore we find that local rigidity implies global rigidity in classical mechanics. In other words, a rigid body (eg. your pen) is rigid only because its small elements are tightly held to their adjacent elements. No action at distance is needed to guarantee that the ends are kept at a fixed distance from each other. This understanding will be our guide when trying to extend the rigidty concept to the relativity theory where global constraints cease to be physical.

2.1 Free Rigid Body Motion

In this short subsection, we will find a geometric representation for the free motion of a classical rigid body. To do so, let us use the CM frame where $\mathbf{v}_0 = \mathbf{0}$ and the motion is purely rotational. The Lagrangian is[1]

$$L = T = \frac{1}{2} \int d\mathbf{r} \rho(\mathbf{r}, t) \Omega_{ni} \Omega_{nj} r_i r_j$$

If $R(t)$ solves $\frac{dR}{dt} = \Omega R$ with the initial condition $R(0) = 1$, then the density is given by

$$\rho(\mathbf{r}, t) = \rho_0\big(R^T(t)\mathbf{r}\big)$$

Here ρ_0 is the mass density corresponding to the initial orientation of the body. Then, the Lagrangian becomes

$$L = \frac{1}{2} M \Omega_{ni} \Omega_{nj} R_{ii'} R_{jj'} \mathcal{J}^0_{i'j'}$$

Here

$$M \equiv \int \rho_0(\mathbf{r}) d\mathbf{r}; \quad \mathcal{J}^0_{ij} \equiv \frac{1}{M} \int d\mathbf{r} \rho_0(\mathbf{r}) r_i r_j$$

Using a parametrization (coordinate system), θ^μ, on $SO(d)$ we may re-write the Lagrangian as

$$L = \frac{1}{2} M \mathcal{J}^0_{ij} \frac{\partial R_{ni}}{\partial \theta^\alpha} \frac{\partial R_{nj}}{\partial \theta^\beta} \frac{d\theta^\alpha}{dt} \frac{d\theta^\beta}{dt}$$

This motivates a metric

$$g_{\alpha\beta} \equiv \mathcal{J}^0_{ij} \frac{\partial R_{ni}}{\partial \theta^\alpha} \frac{\partial R_{nj}}{\partial \theta^\beta}$$

that further simplifies the Lagrangian into

$$L = \frac{1}{2} M g_{\mu\nu} \frac{d\theta^\mu}{dt} \frac{d\theta^\nu}{dt}$$

It is well-known that this Lagrangian induces geodesic paths. In fact, the Euler equation for rigid body dynamics is the same as the geodesic equation on $SO(d)$ with the special metric that we just introduced. We won't be discussing the details any further here.

3 Born Rigidity

Our first *guess* for generalising (1) to relativistic physics would be

$$\nabla_{\{\alpha} V_{\beta\}} = 0$$

but this implies

$$V^\alpha \nabla_\alpha V_\beta = -V^\alpha \nabla_\beta V_\alpha = -\frac{1}{2} \nabla_\beta V^\alpha V_\alpha = 0$$

in other words, all particles move along geodesics. This is not what we wanted. In a rigid body, certain tensions may exist and the particles do not necessarily follow geodesics. To soften this strict condition,

[1] Here we may neglect the interaction potentials that give rise to the rigid constraint. You can say that the atoms are tied together with springs with inifinitely large k values. The potential will go

$$V \sim \frac{F^2}{k}$$

for F being the typical forces and tensions present in the body. This is clearly vanishing for infinite k.

we only insist that for *simultaneous* adjacent space-time points, x^μ and $x^\mu + \delta x^\mu$, the relative velocity vector be orthogonal to the distance vector.

$$\delta x^\mu V_\mu = 0 \Rightarrow \delta x^\mu \delta x^\nu \nabla_\nu V_\mu = 0$$

this means that for some anti symmetric tensro $\Omega_{\alpha\beta}$

$$\nabla_\alpha V_\beta = \Omega_{\alpha\beta} + A_\alpha V_\beta$$

this time, the normalization condition ($V^\mu V_\mu = -1$) gives the following formula for A

$$A_\mu = \Omega_{\mu\nu} V^\nu$$

therefore

$$\nabla_\alpha V_\beta = \Omega_{\alpha\mu}(\delta^\mu_\beta + V^\mu V_\beta)$$

while the projector on the right may suggest that it is impossible to reverse the equation and find Ω in terms of velocity derivatives, the anti symmetry assumption makes this task doable.

$$\Omega_{\alpha\beta} = \nabla_\alpha V_\beta - V_\beta V^\mu \nabla_\mu V_\alpha \tag{2}$$

This is what Born suggested in 1909 as the equation that governs rigid body motion in relativistic physics.[2]

Let us see where this leads us in the simplest case i.e. the 1+1 dimensional Minkowski space-time. Here, the velocity at each point is described by a single real number $\alpha(x,t)$.

$$V^\mu = (\cosh\alpha, \sinh\alpha)$$

the Born rigidity is equivalent to

$$\partial_x \alpha + \tanh(\alpha)\partial_t \alpha = 0$$

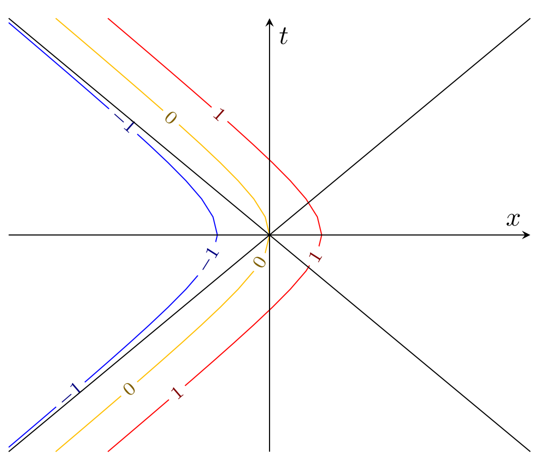
Now consider a finite rigid train sitting on a rail track. For times $t < 0$, the whole train is sitting still.

$$\alpha(x,t) = 0; \quad \forall t \leq 0$$

at $t = 0$, $x = 0$ the driver *decides* to move the train forwards say by exerting a constant force to it. Before proceeding, let us foliate the space-time with timelike curves

$$t(\tau) = \sinh\tau; \quad x(\tau) = \xi + 1 - \cosh\tau$$

1+1 Minkowski Space-Time



In the cooridnate system (τ, ξ), the Born rigidity (2) is written as

$$\frac{\partial \alpha}{\partial \xi}\big(1 + \tanh\alpha \tanh\tau\big) + \frac{\tanh\alpha}{\cosh\tau}\frac{\partial \alpha}{\partial \tau} = 0$$

Now note that the curve corresponding to $\xi = -1$, is causally disconnected from the driver's decision. Therefore, we may write

$$\alpha(\tau, -1) = 0; \ \ \forall \tau$$

But, this quickly implies

$$\alpha(\tau, \xi) = 0; \ \ \forall \tau, \xi$$

In other words, the existence of a non trivial (i.e. not in uniform motion) Born rigid body contradicts the causality principles. In fact, our initial assumption that led to the equations of motion, was not causal to begin with. Even for small distances, the assumption that the velocities are adjusted momentarily will lead to super-luminal signals.

The Born rigidity notion has appeared in many disguises throughout the years. (For example Cf. [3]) and is considered as the most prominent rigidity notion to work with. However, it suffers from causality problems as mentioned above and therefore may not be counted as the answer to the question of what a relativistic rigid object is. In fact, according to [2], it appears that at the time, people were more concerned about the kinematic problems and not the causality problems regarding the rigid motion. Therefore stationary solutions of (2) were used to describe stationary rigid motions and their corresponding geometries.

As an example of such stationary solutions, let us consider the stationary, Born rigid rotation of a disk in $2+1$ dimensions. The line element is

$$ds^2 = -dt^2 + dr^2 + r^2 d\phi^2$$

with non-zero symbols

$$\Gamma^{\phi}_{r\phi} = \Gamma^{\phi}_{\phi r} = \frac{1}{r}; \quad \Gamma^{r}_{\phi\phi} = -r$$

We are looking for a velocity field

$$V^{\mu} = \Big(\cosh\alpha, 0, \frac{1}{r}\sinh\alpha\Big)$$

$$V_{\mu} = \Big(-\cosh\alpha, 0, r\sinh\alpha\Big)$$

with $\alpha = \alpha(r)$. Born rigidity is then equivalent to

$$\frac{d\alpha}{dr} = \frac{\sinh 2\alpha}{2r}$$

integrating the first order equation, we get

$$\tanh\alpha = \omega_0 r$$

note that this velocity field may be written as

$$\vec{V} = \frac{\vec{\partial}_t + \omega_0 \vec{\partial}_\phi}{\sqrt{1 - \omega_0^2 r^2}}$$

which is proportional to a Killing vector field. Below, we will discuss this relationship in more detail below.

3.1 Relationship to Killing Fields and the Herglotz-Noether Theorem

A Killing vector field, ξ^μ is one that satisfies the Killing equation

$$\nabla_{\{\mu}\xi_{\nu\}} = 0$$

For a timelike[2] Killing field, define

$$\lambda \equiv \frac{-1}{2}\log\big(-\xi^\mu\xi_\mu\big)$$

note that $\xi^\alpha\nabla_\alpha\lambda = 0$ is implied by the Killing equation. Now consider the properly normalized velocity field

$$V^\mu \equiv e^\lambda \xi^\mu$$

a simple calculation shows that the following tensor is anti-symetric

$$\Omega_{\alpha\beta} = \nabla_\alpha V_\beta - V_\beta V^\mu \nabla_\mu V_\alpha$$

in other words

Converse Herglotz-Noether Th'm: *Any motion tangent to a (timelike) Killing field is Born rigid.*

To see if the converse also holds, let us seek a λ that guarantees that

$$\xi^\mu \equiv e^{-\lambda}V^\mu$$

is a Killing field. Writing down the Killing equation we find the equivalent condition

$$\nabla_{\{\alpha}V_{\beta\}} = V_{\{\alpha}\nabla_{\beta\}}\lambda$$

If we project this on V^β, use (2), and assume $V^\beta\nabla_\beta\lambda = 0$ (speculated based on our previous results) we get

$$\partial_\alpha\lambda = V^\mu\nabla_\mu V_\alpha$$

which proves the followin lemma.

Lemma: *A Born rigid motion is tangent to some timelike Killing field iff*

$$\nabla_\beta V^\mu \nabla_\mu V_\alpha = \nabla_\alpha V^\mu \nabla_\mu V_\beta$$

In 1910, Herglotz [5] and Noether[3] [6] independently proved that in the 3+1 Minkowski space-time the above condition holds for any Born flow.

Herglotz-Noether Th'm: *In the 3+1 Minkowski space-time, every Born rigid motion is tangent to a timelike Killing field.*

Later, in 1967, Wahlquist and Estabrook generalised this result to conformally flat space-times.[7] However, a full generalization of the theorem is impossible as [8] provides a counter example in Bianchi space-time. Nevertheless, the Herglotz-Noether theorem hints that Born rigidity not only violates the causality principles, but also that it is too restrictive to be considered the final solution.

[2]Later, we will be asking particles to move tangent to this field. This is the only reason we want the Killing field to be timelike.

[3]This is Fritz Noether, Emmy's brother.

There is a puzzling problem here and before *solving* it, this project may not be considered complete. The Born equation in 1+1 Minkowski space-time $\big(\alpha_{,x} + \tanh(\alpha)\alpha_{,t} = 0\big)$ is a well behaved first order partial equation for $\alpha(x,t)$ and must have a unique solution corresponding to any sufficient initial condition. Let $x^\mu(\tau)$ be any timelike worldline, then there must exist a Born flow that has to $V^\mu = dx^\mu/d\tau$ on the worldline. However, such a flow could be considered as a counter example for the Herglotz-Noether theorem!

4 Rigid Fluids

Another approach is to achieve rigidity via considering a very viscous and non-compressible fluid. For a viscid fluid we have

$$T^{\mu\nu} = pg^{\mu\nu} + (p+\rho)V^\mu V^\nu + \eta\nabla^{\{\mu}V^{\nu\}} + \lambda(\nabla_\alpha V^\alpha)g^{\mu\nu}$$

this implies the equation of motion

$$\nabla_\nu p + V^\mu V_\nu \nabla_\mu(p+\rho) + (p+\rho)V^\mu\nabla_\mu V_\nu + (p+\rho)V_\nu\nabla_\mu V^\mu + \eta\Big(\nabla_\mu\nabla^\mu V_\nu + \nabla_\mu\nabla_\nu V^\mu\Big) + \lambda\nabla_\nu\nabla_\mu V^\mu = 0$$

η and λ are called shear and bulk viscosities respectively; the larger they are, the more resistant the fluid will be to shear and bulk deformations. Being interested in rigidity, we consider the limit $\eta, \lambda \to \infty$. In this limit, the equations of motion become

$$\Box V^\mu + \mathcal{R}^\mu_\nu V^\nu + (1 + \frac{\lambda}{\eta})\nabla^\mu\nabla_\alpha V^\alpha = 0$$

Now we speculate that for a rigid fluid, the flow is incompressible and therefore $\nabla_\alpha V^\alpha = 0$. This gives

$$\Box V^\mu + \mathcal{R}^\mu_\nu V^\nu = 0 \tag{3}$$

This does not suffer from the causality problem since it looks exactly the same as the equation for the electromagnetic four potential in vacuum. However, the solutions do not necessarily satisfy $\nabla_\alpha V^\alpha = 0$ which we used as a premise to derive this equation of motion. Even worse, being of second order, this does not guarantee that the normalization condition continues to hold.

One way to overcome this, as suggested by [4] is to use a perfect fluid with maximum physical speed of sound.

$$T^{\mu\nu} = \rho\big(g^{\mu\nu} + 2V^\mu V^\nu\big)$$

This leads to

$$V^\alpha\nabla_\alpha V_\mu - V_\mu\nabla_\alpha V^\alpha + \frac{1}{2}\nabla_\mu \log\rho = 0$$

While this may be useful in 1+1 dimensions, in higher dimensions the equation is not restrictive enough and does not produce any resistance against shear deformations. With this remark we conclude our search for a rigid fluid and try out another idea.

5 Rigid Solids

In this last section, we will be following [9] in developing a relativstic theory of elastic solids in order to find the physical criteria for a rigid motion. However, our choice for the *right* set of parameters that represent a rigid solid will be different.

Let us begin with labelling the particles in a solid with coordinates y^a with $a = 1, 2, \cdots, n$. Then, the motion is fully described by determining the scalar fields $y^a(x^\mu)$. For what we will be doing shortly, this will be more convenient than dealing with the velocity field. This viewpoint already suggests a field theoretic approach and therefore we will be seeking a Lagrangian model to describe the motion.

$$\mathcal{L} = \mathcal{L}(y^a, \partial_\mu y^a)$$

higher order derivatives are suppressed since we are looking for a second order equation of motion (for y^a) similar to classical elastic models. Also, we assume homogeneity to drop any explicit dependence of $\mathcal{L}$ on y^a.

At any point in the space-time, the four velocity is found by solving

$$V^\mu \partial_\mu y^a = 0; \quad \forall a$$

this allows us to locally write the line element as

$$ds^2 = -d\tau^2 + h_{ab} dy^a dy^b$$

with the understanding that $V = \frac{\partial}{\partial \tau}$. The spatial metric is given by

$$h_{ab} = g_{\mu\nu} \frac{\partial x^\mu}{\partial y^a} \frac{\partial x^\nu}{\partial y^b}; \quad h^{ab} = g^{\mu\nu} \frac{\partial y^a}{\partial x^\mu} \frac{\partial y^b}{\partial x^\nu}$$

We assume that in the absence of any tension, the distances between particles are given by a metric k

$$ds^2 = k_{ab} dy^a dy^b$$

Then a solid is called elastic if the Lagrangian depends only on the relative metric $h/k = \sqrt{k^{-1}} h \sqrt{k^{-1}}$. For simplicity, we assume that the original metric k is Euclidean around the special y^a that we are considering, i.e.

$$k_{ab} = \delta_{ab}$$

This assumption costs no generality but allows us to write $\mathcal{L} = \mathcal{L}(h)$. The energy momentum tensor is

$$T_{\mu\nu} = \mathcal{L} g_{\mu\nu} - 2 \frac{\partial \mathcal{L}}{\partial g^{\mu\nu}} = \mathcal{L} g_{\mu\nu} - 2 \frac{\partial \mathcal{L}}{\partial h^{ab}} \frac{\partial y^a}{\partial x^\mu} \frac{\partial y^b}{\partial x^\nu}$$

in the local comoving coordinates introduced above, this has $\rho = T_{\tau\tau} = -\mathcal{L}$. In other words the Lagrangian is given by the density in terms of h^{ab}. If we further assume isotropy, then the density would only depend on the metric eigenvalues.

$$ds^2 = -d\tau^2 + \sum_a \left(s_{(a)} dy^a \right)^2; \quad \rho = \rho(s_a)$$

For example, let us consider the special case that ρ depends only on the determinant of h.

$$T_{\mu\nu} = -\rho g_{\mu\nu} + 2 \frac{d\rho}{d \det[h^{\cdot\cdot}]} \det[h^{\cdot\cdot}] h_{ab} \frac{\partial y^a}{\partial x^\mu} \frac{\partial y^b}{\partial x^\nu}$$

$$= -\rho g_{\mu\nu} + 2 \frac{d\rho}{d \det[h^{\cdot\cdot}]} \det[h^{\cdot\cdot}] \big(g_{\mu\nu} + V_\mu V_\nu \big)$$

$$= p g_{\mu\nu} + (p + \rho) V_\mu V_\nu$$

with

$$p \equiv -\rho + 2\frac{d\rho}{d\det[h^{\cdot\cdot}]}\det[h^{\cdot\cdot}]$$

which clearly describes a perfect fluid model.

To define the rigid solid, we consider a Hookean model where the density is given by a quadratic formula in terms of small deformation vectors. For small displacements, $y^a \to y^a + \xi^a$, the eigenvalues become $s_a = 1 + \sigma_a$ with $\sigma = \mathcal{O}(\xi)$. Therefore a second order (Hookean) model may be written in the form

$$\rho(\sigma_a) = \rho_0 + \sum_a A_a\sigma_a + \sum_{ab} B_{ab}\sigma_a\sigma_b$$

permutation symmetry then implies

$$\rho(\sigma_a) = \rho_0 + A_1\sum_a \sigma_a + A_2\sum_a \sigma_a^2 + A_3\Big(\sum_a \sigma_a\Big)^2$$

Dropping the constant term, we may write the Lagrangian as

$$\mathcal{L} = A_1\sum_a \sigma_a + A_2\sum_a \sigma_a^2 + A_3\Big(\sum_a \sigma_a\Big)^2$$

It remains to choose the right set of parameters that give rise to a rigid solid. In order to do that, let us put start with our solid at rest in a flat space-time and see how small perturbations evolve over time. The mathematical set up is

$$y^a = x^a + \xi^a(x)$$

with small ξ. We will need to carry out the calculations up to $\mathcal{O}(\xi^2)$.

5.1 The Spatial Metric

Let us begin with the velocity field given by

$$V^\mu = \Big(1 + \frac{1}{2}\sum_a(\partial_t\xi^a)^2, -\partial_t\xi^a + \sum_b(\partial_t\xi^b)(\partial_b\xi^a)\Big)$$

Next we have to do the spatial metric; that is $ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu$ for

$$dx^\mu V_\mu = 0; \quad dx^\mu\partial_\mu y^a = dy^a$$

A careful calculation then gives

$$dx^0 = \sum_a dy^a\Big\{-\partial_t\xi^a + \sum_b(\partial_t\xi^b)\big(\partial_a\xi^b + \partial_b\xi^a\big)\Big\}$$

$$dx^a = \sum_b dy^b\Big\{\delta^a_b + \partial_b\xi^a + (\partial_t\xi^b)(\partial_t\xi^a) + \sum_c(\partial_b\xi^c)(\partial_c\xi^a)\Big\}$$

this leads to a spatial metric

$$h_{ab} = \delta_{ab} - \partial_b\xi^a - \partial_a\xi^b + (\partial_t\xi^a)(\partial_t\xi^b) + \sum_c\big[(\partial_a\xi^c)(\partial_c\xi^b) + (\partial_b\xi^c)(\partial_c\xi^a) + (\partial_a\xi^c)(\partial_b\xi^c)\big] \quad (4)$$

5.2 Longitudinal Waves

The longitudinal waves are best discussed in 1+1 dimensional Minkowski space-time. Let

$$y = x + \xi(t, x)$$

then the longitudinal speed of sound in this material will be

$$c_L^2 = -\frac{3A_1 + 4(A_2 + A_3)}{A_1} \tag{5}$$

5.3 Transverse Waves

This time, we work in the $2+1$ dimensional Minkowski space-time. Let

$$y^1 = x^1; \quad y^2 = x^2 + \xi(t, x^1)$$

this yields a transverse speed of sound

$$c_T^2 = -\frac{A_1 + 2A_2}{A_1} \tag{6}$$

5.4 The Sigma Model Solid

The so called (by [9]) sigma model solid is achieved by maximizing the longitudinal and transverse propagation speeds. According to (5) and (6) this means

$$A_2 = -A_1; \quad A_3 = 0$$

For this choice of parameters, the Lagrangian is equivalent to that of a linear sigma model

$$\mathcal{L} = \sum_a (\partial_\mu \xi^a)(\partial^\mu \xi^a) \tag{7}$$

However, this equivalence is only valid for small oscillations and should not be over emphasized.

Although the sigma model solid suffers from no causal or kinematic problems, [9] points out that this model leads to negative bulk modulus and therefore puts it aside as an inconsistent model. A more serious problem with the elastic solid models is that in general, the lagrangian will not be of second order in displacement vector ξ^a. This means that the propagation speed for both transverse and longitudinal waves may be different in the nonlinear regime.

6 Conclusion

In this paper we have reviewed several candidates for a relativistic rigid body and concluded that none of them deserves to be regarded as the final answer since they all suffer from physical inconsistencies. The diversity of the proposals however, suggests that it may be possible to find a consistent model with maximal wave propagation speed.

Also, a geometric interpretation for the free rotation of a classical rigid body is provided in section 2. This may be used to analyse the dynamical features of the free precession such as stability or ergodicity.

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