QFT, Done Right

Koorosh Sadri

October 1, 2024

Contents

Notations and Conventions

• $\hbar = \mu_0 = \varepsilon_0 = k_B = 4\pi G = 1$.

• The space-time metric has $-+++\cdots$ signature. Your height is a real number and the distance between your house today and your house tomorrow is imaginary!

• The Fourier transforms look as below

$$\tilde{f}(k_1, k_2, \cdots, k_n) \equiv (2\pi)^{-n/2} \int d\mathbf{x}\, f(x_1, x_2, \cdots, x_n) \exp\big[-i(k_1 x_1 + \cdots + k_n x_n)\big]$$

$$f(x_1, x_2, \cdots, x_n) = (2\pi)^{-n/2} \int d\mathbf{k}\, \tilde{f}(k_1, k_2, \cdots, k_n) \exp\big[+i(k_1 x_1 + \cdots + k_n x_n)\big]$$

• Hilbert space operators DO NOT wear hats.

• Space-time dimension is $n+1$ unless otherwise specified.

• Spatial vectors are denoted by boldface characters like $\mathbf{P}$, etc. Whereas the space-time vectors can be recognized from their accompanying Greek indices eg: P^α, etc.

• Complex conjugate of any qunatity Q is Q^* and the Hermitian conjugate will be $Q^\dagger$ (whenever applicable)

• Repeated double indices are always summed over unless there are 3 (or more) of them or one of them is protected by parentheses:

$$A_i B_i, \quad A_\mu B^\mu, \quad A_{ii} B_i, \quad A_i B_{(i)}$$

• Tuples that are not spatial or space-time vectors are denoted by a sub-tilde; eg. $\underset{\sim}{p}$ for $\{p_i\}$ the set of momenta.

• To distinguish between different pictures when dealing with observables, look at the arguments. If there is no time dependence, we are working in the Schroedinger picture, if there is a time dependence, we are working in the Heisenberg picture and finally, if there is a superscript 0 and a time dependence, we are working in the interaction picture.

• An asterisk (*) in the title of a section means that it is still under construction/correction.

Part I

Classical Mechanics

Chapter 1

A Review of Lagrangian and Hamiltonian Mechanics

1.1 * Dynamical Systems, What to expect

A continuous dynamical system is a tuple $(\mathtt{X}, \Omega)$. The first entry, $\mathtt{X}$ is the set of *observables* which are real random variables. It is assumed that the set of observables satisfies

- $f(X) \in \mathtt{X} \qquad \forall\, f : \mathbb{R} \to \mathbb{R},\ X \in \mathtt{X}$
- $\alpha X_1 + \beta X_2 \in \mathtt{X} \qquad \forall\, \alpha, \beta \in \mathbb{R}\, X_1, X_2 \in \mathtt{X}$

The second entry Ω is a convex set of states denoted by ω. A state $\omega \in \Omega$ is a single parameter map from the set of observables $\mathtt{X}$ to the set of probability measures on $\mathbb{R}$, where the single parameter is called (dynamical) *time*. The practical interpretation is that if the system is in the state ω and one measures the observable X at some time t, the result will be a real random variable with distribution $\omega(X)$. This motivates us to assume that the following consistency conditions hold

- $\omega_t(f(X)) = f * \omega_t(X) \qquad \forall\, f : \mathbb{R} \to \mathbb{R},\, \omega \in \Omega,\ X \in \mathtt{X}, t \in \mathbb{R}$
- $\left[p\omega + (1-p)\omega'\right]_t(X) = p\omega_t(X) + (1-p)\omega'_t(X)$

We will generally limit our discussion to systems where it is possible to come up with maps $\varphi^{\text{Sch.}}$ and $\varphi^{\text{Hei.}}$ such that the following identities hold.

$$\omega_t(X) = \varphi^{\text{Sch.}}_{s\to t}(\omega)_s(X)$$
$$\omega_t(X) = \omega_s\left(\varphi^{\text{Hei.}}_{s\to t}(X)\right)$$

1.2 Lagrangian Systems

A *classical* mechanical system consists of a number of degrees of freedom, q_i for i in some index set $\mathcal{I}$, and a Lagrangian function L. The Lagrangian is assumed to be a function of the degrees of freedom q_i, their first order time derivatives and possibly of time itself. The equations of motion are derived by optimizing the action

$$S = \int L(q_i, \dot{q}_i; t)\, dt$$

This leads to the Euler-Lagrange equations

$$\frac{\partial L}{\partial q_i} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}$$

The equations are clearly invariant under a transformation

$$L \to L + \frac{dX}{dt}$$

for some $X = X(q_i, \dot{q}_i; t)$. This motivates us to call any transformation $q_i \to q_i'$ a symmetry of the system if it leads to a change in the Lagrangian that has the above form. An infinitesimal such transformation looks like below

$$q_i \to q_i + \varepsilon \tau_i(q_j, \dot{q}_j)$$

which in turn results in a change in the Lagrangian

$$L \to L + \varepsilon\big(\tau_i \frac{\partial L}{\partial q_i} + \dot{\tau}_i \frac{\partial L}{\partial \dot{q}_i}\big)$$

$$= L + \varepsilon\Big[\tau_i\big(\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}\big) + \frac{d}{dt}\big(\tau_i \frac{\partial L}{\partial \dot{q}_i}\big)\Big]$$

$$= L + \varepsilon \frac{d}{dt}\big(\tau_i \frac{\partial L}{\partial \dot{q}_i}\big)$$

On the other hand we assumed that $L \to L + \varepsilon \frac{dX}{dt}$.[1] Comparing the two results, we find a conservation law

$$\boxed{\frac{d}{dt}\Big(\tau_i \frac{\partial L}{\partial \dot{q}_i} - X\Big) = 0}$$

This result is famously known as the Noether's theorem.

As a first example, consider a time shift in the degrees of freedom $q_i(t) \to q_i(t+\varepsilon) \approx q_i(t) + \varepsilon \dot{q}_i(t)$. For a Lagrangian with no explicit time dependence, this results in

$$L \to L + \varepsilon \dot{L}$$

and therefore the time shift will be asymmetry of the system with $X = L$. The corresponding conserved charge will be defined to be the Hamiltonian of the system

$$H = \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L$$

It is customary to define the conjugate momenta as

$$p_i \equiv \frac{\partial L}{\partial \dot{q}_i}$$

then the Hamiltonian may be regarded as the Legendre transformation of the Lagrangian with respect to velocities $\dot{q}_i$ and therefore a function of p_i and q_i

$$H(q_i, p_i) = \dot{q}_j p_j - L$$

[1] There is a subtle difference betweenequations $\delta L = \varepsilon X$ and $\delta L = \varepsilon \tau_i \partial L / \dot{q}_i$. The former is assumed to hold as an identity resulting from the specific properties of the Lagrangian function and has nothing to do with dynamics; however, the latter holds only when the Euler-Lagrange equations are satisfied. This is sometimes called ”on-shell”.

this new function allows us to write the equations of motion as first order differential equations over the phase space (q_i, p_i)

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}; \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}$$

The Hamiltonian equations of motion allow us to write down the time derivatives of different phase space functions in a specific format

$$\begin{aligned}\frac{d}{dt} f(q_i, p_i; t) &= \frac{\partial f}{\partial t} + \frac{\partial f}{\partial q_j}\frac{\partial H}{\partial p_j} - \frac{\partial f}{\partial p_j}\frac{\partial H}{\partial q_j} \\ &= \frac{\partial f}{\partial t} + \{f, H\}\end{aligned}$$

Where in the last line, we have introduced the Poisson brackets

$$\{A, B\} \equiv \frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i}$$

The above equations, reveal yet another relation between the Hamiltonian and time shifts since the Hamiltonian is generating a time shift. Does such a relation also exist between any other conserved charge Q and its corresponding symmetry transformation? To check this, let us study the infinitesimal transformation generated by some conserved charge

$$\begin{aligned} q_i \to q_i + \varepsilon\{q_i, Q\} &= q_i + \varepsilon \frac{\partial Q}{\partial p_i} \\ &= q_i + \varepsilon\Big[\tau_i + \frac{\partial^2 H}{\partial p_i \partial p_j}\big(p_k \frac{\partial \tau_k}{\partial \dot{q}_j} - \frac{\partial X}{\partial \dot{q}_j}\big)\Big]\end{aligned}$$

On the other hand, the symmetry condition asks the following to hold as an identity, on and off shell

$$\frac{\partial X}{\partial t} + \frac{\partial X}{\partial q_i}\dot{q}_i + \frac{\partial X}{\partial \dot{q}_i}\ddot{q}_i = \frac{dX}{dt} = \delta L = \frac{\partial L}{\partial q_i}\tau_i + \frac{\partial L}{\partial \dot{q}_i}\frac{\partial \tau_i}{\partial t} + \frac{\partial L}{\partial \dot{q}_i}\frac{\partial \tau_i}{\partial q_j}\dot{q}_j + \frac{\partial L}{\partial \dot{q}_j}\frac{\partial \tau_j}{\partial \dot{q}_i}\ddot{q}_i$$

Since none of the functions depend on second order time derivatives, we may equate the corresponding coefficients to get

$$\frac{\partial X}{\partial \dot{q}_i} = p_j \frac{\partial \tau_j}{\partial \dot{q}_i}.$$

This in turn implies

$$q_i + \varepsilon\{q_i, Q\} = q_i + \varepsilon \tau_i$$

In other words, not only every symmetry gives rise to a conserved charge (Noether's theorem), but also every conserved charge generates a symmetry transformation via Poisson brackets (Converse of Noether's theorem).

Assuming a system with n degrees of freedom has c independent, continuous conserved charges means that its generic integral curve will cover some $2n - c$ dimensional manifold in the phase space. For instance, the Keplerian system has 3 degrees of freedom and 5 conserved charges (Energy, angular momentum vector and the Laplace-Runge-Lenz vector) therefore all Keplerian orbits are closed on themselves.

As an example for the converse, consider the Lagrangian describing two independent harmonic oscillators.

$$L = \frac{1}{2}\Big(\dot{q}_1^2 + \dot{q}_2^2 - q_1^2 - \lambda^2 q_2^2\Big)$$

Regardless of the specific value of λ, the Lagrangian is covariant under *independent* time shifts and therefore the following are conserved

$$H_1 = \frac{1}{2}\Big(p_1^2 + q_1^2\Big); \quad H_2 = \frac{1}{2}\Big(p_2^2 + \lambda^2 q_2^2\Big)$$

The remaining 2-dimensional manifold is a torus; let us parametrize it as

$$p_1 = \sqrt{2H_1}\cos\phi_1; \;\; q_1 = \sqrt{2H_1}\sin\phi_1; \;\; p_2 = \sqrt{2H_2}\cos\phi_2; \;\; q_2 = \lambda^{-1}\sqrt{2H_2}\sin\phi_2$$

The integral curves are described by

$$H_1 = E_1; \;\; H_2 = E_2; \;\; \phi_1 = t + \alpha_1; \;\; \phi_2 = \lambda t + \alpha_2$$

For irrational λ, this densely covers the torus and therefore no other conserved charge *could* exist; however, for $\lambda = n_1/n_2$ the curves are 1 dimensional and therefore another conserved charge *must* exist. Indeed the charge is

$$Z = \exp\big[2\pi i(n_1\phi_1 - n_2\phi_2)\big]$$

More often than not, it is the case that we want to work with a statistical/probabilistic description of a mechanical system. This is achieved by having access to a *consistent* set of expected values for all phase space observables $\langle f(q_i, p_i; t)\rangle_t$. However, it is much more convenient to work with a probability density function ρ that produces all the expected values. It is natural to assign the time evolution to the density ρ. We will call this the Schroedinger picture.

$$\langle f(q_i, p_i; t)\rangle_t = \int d\underset{\sim}{p}\, d\underset{\sim}{q}\; f(q_i, p_i; t)\rho(q_i, p_i, t); \qquad \frac{\partial\rho}{\partial t} = -\{\rho, H\}$$

Note that the minus sign is what distinguishes ρ from an observable. It is also possible to assign the time evolution to the observables and consider ρ as being a constant measure in time. This is called the Heisenberg picture.

$$\langle f(q_i, p_i; t)\rangle_t = \int d\underset{\sim}{p}\, d\underset{\sim}{q}\; f_t(q_i, p_i)\rho(q_i, p_i); \qquad \frac{df_t}{dt} = \frac{\partial f}{\partial t} + \{f_t, H\}$$

1.3 Hamilton's Function

Let's assume that the system starts from a configuration $q_{0,i}$ at an initial time t_0 and ends up in the configuration $q_{1,i}$ at a final time t_1 all through a solution $q(t)$. The action can be calculated as a function of these 4 boundary values

$$\mathscr{S}(t_0, q_{0,i}, t_1, q_{1,i}) = \int_{t_0}^{t_1} dt\, L(t, q(t), \dot{q}(t))$$

A slight change in the boundary values $q_{0,i}$ and $q_{1,i}$ gives

$$\delta\mathscr{S} = \frac{\partial\mathscr{S}}{\partial q_{0,i}}\delta q_{0,i} + \frac{\partial\mathscr{S}}{\partial q_{1,i}}\delta q_{1,i} = \int_{t_0}^{t_1} dt\delta L = p_{1,i}\delta q_{1,i} - p_{0,i}\delta q_{0,i}$$

Leading to

$$\frac{\partial\mathscr{S}}{\partial q_{1,i}} = p_{1,i}$$

Continuing the path further in time gives

$$\delta\mathscr{S} = \frac{\partial\mathscr{S}}{\partial t_1}\delta t_1 + p_{1,i}\dot{q}_{1,i}\delta t_1 = L_1\delta t_1$$

This yields the time derivative as

$$\frac{\partial\mathscr{S}}{\partial t_1} = -H_1$$

1.4 * Constrained Systems

In this section we discuss how peculiarities in the Lagrangian/Hamiltonian function may be used to describe constrained systems. This will be useful when we try to quantise systems with gauge symmetries. At first we will try to do the entire analysis without referring to the Hamiltonian formalism. The Dirac theory concerning the Hamiltonian treatment of constrained systems is then presented.

1.4.1 The Lagrangian Formalism

In the previous parts, we always assumed that the Euler-Lagrange equations of motion, derived via optimizing the action integral, are always enough to determine the accelerations, $\ddot{q}_i$, in terms of the positions, q_i, and the velocities, $\dot{q}_i$. That is

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i} \;\Rightarrow\; \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}\ddot{q}_j + \frac{\partial^2 L}{\partial q_j \partial \dot{q}_i}\dot{q}_j = \frac{\partial L}{\partial q_i} \;\Rightarrow\; \ddot{q}_i = A_i(q_j, \dot{q}_j)$$

However, this is only possible if the matrix

$$M_{ij} \equiv \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}$$

is not singular. It is not even a serious problem if the matrix is singular on a measure zero surface in the $2N$ dimensional space of $(q_i, \dot{q}_j)$; the equations of motion may still be used arbitrarily near the surface, then gluing the solutions together before and after passing through the singular surface will usually be considered as the answer to the Lagrangian problem. (Cf. exercises)

The interesting case (in this section) happens when the matrix M is singular over a whole domain. At the first glance it seems that in such cases, the accelerations do not exist unless a number of constraints are satisfied on $(q_i, \dot{q}_i)$ and even then, they are not uniquely determined. To see what the constraints are, let us define

$$F_i(\underset{\sim}{q}, \underset{\sim}{\dot{q}}) \equiv \frac{\partial L}{\partial q_i} - \frac{\partial^2 L}{\partial q_j \partial \dot{q}_i}\dot{q}_j$$

Then, M's null eigen-vectors $z_i^a(\underset{\sim}{q}, \underset{\sim}{\dot{q}})$ satisfy $z_i^a M_{ij} = 0$; therefore, Lagrange's equations of motion admit solutions only if the following first order constraints are satisfied

$$z_i^a F_i = 0$$

Note that such constraints need to be satisfied even on the boundary/initial conditions; otherwise the action will have no optimal solution. Out of the constraints above, only a subset of them depend on the velocities in an essential way. If the matrix $\left[\frac{\partial (z_i^a F_i)}{\partial \dot{q}_j}\right]$ is singular, a number of the constraints may be separated to form

$$\text{constraints:} \begin{cases} \gamma^a(\underset{\sim}{q}) = 0 & \text{type-A constraints} \\ \gamma^b(\underset{\sim}{q}, \underset{\sim}{\dot{q}}) = 0 & \text{type-B constraints} \end{cases}$$

The constraints above, may further decrease the functional rank of the M_{ij} matrix, thereby adding even more constraints. On the other hand, the constraints not only need to be satisfied at a single moment in time, but also through the whole path, this means we want to have $\dot{\gamma}^a = 0$ and $\dot{\gamma}^b = 0$. The first set leads to more type-B constraints while the second set, give rise to new equations for the accelerations, $\ddot{q}$. Solving the full problem, involves adding these implications and repeating until one reaches a state with the equations

$$\text{constraints:} \begin{cases} \gamma^a(\underset{\sim}{q}) = 0 & \text{type-A constraints} \\ \gamma^b(\underset{\sim}{q}, \underset{\sim}{\dot{q}}) = 0 & \text{type-B constraints} \end{cases} \quad ; \quad M\underset{\sim}{\ddot{q}} = \underset{\sim}{F} \quad ; \quad N_i^c \ddot{q}_i = G_i$$

where

- The null vectors from the acceleration equations, don't add anything to the constraints.
- The time derivative of type-A constraints don't add anything to the type-B constraints.
- The type-B constraints essentially depend on the velocities; that is the matrix $\frac{\partial \gamma^b}{\partial \dot{q}_i}$ is full-rank.
- The time derivative of the type-B constraints does not add anything to the acceleration equations.
- The constraints do not further decrease the rank (increase the singularity) of the acceleration equations.

Even at this final stage, the acceleration equations may not be enough to provide a unique solution, in that case, the dynamical paths that give rise to the optimal action are not unique. It may also be the case that these equations, lead to inconsistencies, in those cases, there is no optimal solution for the action and the Lagrangian method fails.

1.4.2 The Hamiltonian Formalism: Dirac's Theory

Whenever the Lagrange equations of motion fail to determine the accelerations, a similar problem arises in the Hamiltonian formalism and we can not write all the $\underset{\sim}{\dot{q}}$s in terms of the momenta. In fact the necessary and sufficient condition for this process is given via the implicit function theorem as

$$\det\left(\frac{\partial p_i}{\partial \dot{q}_j}\right) = \det\left(\frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}\right) = \det(M_{ij}) \neq 0$$

If this is not satisfied, then there are primary (i.e. off shell) constraints connecting the momenta and coordinates.

$$\phi_\alpha(\underset{\sim}{p}, \underset{\sim}{q}) = 0$$

Let N denote the number of degrees of freedom in our system and R the functional rank of the mass matrix M_{ij}; then $N - R$ will be the number of such primary constraints. Of course at this stage (before writing the equations of motion) the coordinates $\underset{\sim}{q}$ are all independent and therefore it would be impossible to use these primary constraints to find type-A constraints on the coordinates; this means that the matrix

$$\left(\frac{\partial \phi_\alpha}{\partial p_i}\right)$$

is full-rank. This allows us to re-write the constraints as

$$p_i - \psi_i(\underset{\sim}{q}, p_1, \cdots, p_R) = 0\ ;\ \ i \in \{R+1, \cdots, N\}$$

This all means that the canonical variables $(\underset{\sim}{p}, \underset{\sim}{q})$ can not serve as a coordinate system for the $2N$ dimensional space of positions and velocities spanned by $(\underset{\sim}{q}, \underset{\sim}{\dot{q}})$. While the former is traditionally called the phase space, we shall call the latter by the name *Lagrange's space*. The phase space contains non-physical points that do not correspond to any set of positions and velocities; in other words, the physical systems are constrained to a sub-manifold of the phase space described by the primary constraints. On the other hand, if we add $N-R$ of the (carefully chosen) velocities as auxiliary coordinates to the $\underset{\sim}{q}$ and the independent momenta, we may construct a healthy coordinate system for the Lagrange's space. In this coordinate system, we can describe the Hamiltonian as

$$H(\underset{\sim}{p},\underset{\sim}{q}) \equiv \sum_{i=1}^{R} p_i\dot{q}_i + \sum_{i=R+1}^{N} \psi_i\dot{q}_i - L$$

where some of the $\dot{q}_i$'s are independent coordinates and some are functions of other coordinates. Note that this notation implies that the Hamiltonian is only a function of the canonical variables; to see this, we can differentiate both sides while keeping the canonical variables constant.

$$dH_{\underset{\sim}{p},\underset{\sim}{q}} = \sum_i d\dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - \sum_i d\dot{q}_i \frac{\partial L}{\partial \dot{q}_i} = 0$$

Then we can differentiating the Hamiltonian definition with respect to p_i with $i \le R$, to get

$$\frac{\partial H}{\partial p_i} = \dot{q}_i + \sum_{j=R+1}^{N} \dot{q}_j \frac{\partial \psi_j}{\partial p_i}$$

Interestingly, these relations set the auxiliary (undetermined) velocities to be those that correspond to degenerate momenta; i.e. $\dot{q}_i$ with $i > R$.

Now that we know the coordinate systems and the spaces they span, let us focus on the dynamics of the system. In Hamiltonian formalism, we are interested in finding a Hamiltonian flow on the phase space. We have already found expressions for $\dot{q}_i$ with $i \le R$. The time derivative of the momenta are given by the Euler-Lagrange equations

$$\begin{aligned}\dot{p}_i = \frac{\partial L}{\partial q_i} &= \sum_{j\le R} \frac{\partial p_j}{\partial q_i}\dot{q}_j + \sum_{j>R} \frac{\partial \psi_j}{\partial q_i}\dot{q}_j + \sum_{j>R} \frac{\partial \psi_j}{\partial p_k}\frac{\partial p_k}{\partial q_i}\dot{q}_j - \sum_{j\le R} \frac{\partial H}{\partial p_j}\frac{\partial p_j}{\partial q_i} - \frac{\partial H}{\partial q_i} \\ &= \sum_{j\le R} \frac{\partial p_j}{\partial q_i}\Big(\frac{\partial H}{\partial p_j} - \sum_{k>R} \dot{q}_k \frac{\partial \psi_k}{\partial p_j}\Big) + \sum_{j>R} \frac{\partial \psi_j}{\partial q_i}\dot{q}_j + \sum_{j>R} \frac{\partial \psi_j}{\partial p_k}\frac{\partial p_k}{\partial q_i}\dot{q}_j - \sum_{j\le R} \frac{\partial H}{\partial p_j}\frac{\partial p_j}{\partial q_i} - \frac{\partial H}{\partial q_i} \\ &= -\frac{\partial H}{\partial q_i} + \sum_{j>R} \dot{q}_j \frac{\partial \psi_j}{\partial q_i}\end{aligned}$$

It remains to find the $i > R$ velocities in the phase space or prove them irrelevant; this is the focus of what follows.

Dirac's Theory

Let U denote the physical sub-manifold of the phase space where the following primary constraints are satisfied.

$$\phi_i \equiv p_i - \psi_i(q_1, \cdots, q_N, p_1, \cdots, p_R) = 0\ ;\ i > R$$

Since the constraints need to be satisfied at all times in order for the system to remain physically meaningful, we need the time derivative of the constraints to vanish as well. For a generic function of the canonical variables, we have

$$\begin{aligned}\frac{df}{dt} &= \frac{\partial f}{\partial p_i}\Big(-\frac{\partial H}{\partial q_i} + \sum_{j>R} \dot{q}_j \frac{\partial \psi_j}{\partial q_i}\Big) + \sum_{i\le R} \frac{\partial f}{\partial q_i}\Big(\frac{\partial H}{\partial p_i} - \sum_{j>R} \dot{q}_j \frac{\partial \psi_j}{\partial p_i}\Big) + \sum_{j>R} \dot{q}_j \frac{\partial f}{\partial q_j} \\ &= \{f, H\} + \sum_{i>R} \dot{q}_i \Big(\frac{\partial \psi_i}{\partial q_j}\frac{\partial f}{\partial p_j} - \frac{\partial \psi_i}{\partial p_j}\frac{\partial f}{\partial q_j} + \frac{\partial f}{\partial q_i}\Big)\end{aligned}$$

$$= \{f, H\} + \sum_{i>R} \dot{q}_i \{f, \phi_i\} \tag{1.1}$$

For the constraints then we have

$$\frac{d\phi_i}{dt} = \{\phi_i, H\} + \sum_{j>R} \dot{q}_j \{\phi_i, \phi_j\}$$

which leads to equations of motion

$$\sum_{j>R} \dot{q}_j \{\phi_i, \phi_j\} = -\{\phi_i, H\}$$

If the (anti-symmetric) matrix $\{\phi_i, \phi_j\}$ is full-rank on the physical submanifold, then we are done.

$$\dot{q}_i = -\sum_{j>R} C_{ij} \{\phi_j, H\}$$

and therefore the dynamics is given by Dirac's equations of motion

$$\boxed{\frac{df}{dt} = \{f, H\} - \sum_{i,j>R} \{f, \phi_i\} C_{ij} \{\phi_j, H\}}$$

Otherwise, there are secondary constraints corresponding to any null eigenvector of the $\{\phi_i, \phi_j\}$ matrix

$$\lambda_{ai} \{\phi_i, \phi_j\} = 0 \quad \Rightarrow \quad \chi_a \equiv \lambda_{ai} \{\phi_i, H\} = 0$$

these could lead to further secondary constraints by reducing the functional rank of the constraints matrix. Other than that, it would also lead to new equations of motion. The final state is achieved as

$$\text{Auxiliary Equations of Motion} \begin{cases} \text{Primary:} & \{\phi_i, \phi_j\}\dot{q}_j + \{\phi_i, H\} = 0 \\ \\ \text{Secondary:} & \{\chi_a, \phi_i\}\dot{q}_i + \{\chi_a, H\} = 0 \end{cases}$$

$$\text{Constraints} \begin{cases} \text{Primary:} & \phi_i = 0 \\ \\ \text{Secondary:} & \chi_a = 0 \end{cases}$$

when

- Possible singularities in the auxiliary eom's do not add new constraints.

- Time derivative of the constraints, do not add anything to the eom's.

Once again, if the equations of motion are still not enough to determine the dynamics, then the solution may not exists or not be unique.

Note that in essence, our struggle to find $\dot{q}_i$ for $i > R$ on the physical submanifold is equivalent to *correcting* the Hamiltonian as

$$H \to H_0 + \sum_{i>R} u_i \phi_i$$

where $u_i = \dot{q}_i$ are chosen in a way to keep the flow physical at all times. For deterministic systems, the u_i will all be determined, but sometimes, u_i may have some degree of arbitrariness in them.

As our first example, let us consider a particle that is constrained to move on a manifold inside the Euclidean space. The Lagrangian is

$$L = \frac{1}{2}m\dot{\mathbf{x}}^2 - \sum_\alpha q_\alpha f_\alpha(\mathbf{x}) - U(\mathbf{x})$$

The primary constraints read $\phi_\alpha = p_\alpha = 0$ and therefore the Hamiltonian in general is

$$H = \frac{\mathbf{p}^2}{2m} + U(\mathbf{x}) + \sum_\alpha q_\alpha f_\alpha(\mathbf{x}) + \sum_\alpha p_\alpha u_\alpha$$

1.5 Examples of Constrained sytems

1.5.1 Auxiliary Time

As a canonical example of our theory of constrained systems, let's consider a mathematical trick that takes the time t to be a dynamical variable changing through some auxiliary time $\mathtt{t}$ along with the other degrees of freedom $q_i(\mathtt{t})$. The Lagrangian is found by rewriting the action

$$S = \int \tilde{L} dt = \int d\mathtt{t}\ \dot{t} L(q_i, \frac{\dot{q}_i}{\dot{t}_i}, t) = \int d\mathtt{t}\ \tilde{L}(q_i, \dot{q}_i, \dot{t}, t)$$

where the dot here denotes differentiation with respect to $\mathtt{t}$. The momenta are given by

$$\tilde{p}_i = \frac{\partial \tilde{L}}{\partial \dot{q}_i} = \frac{\partial L}{\partial \dot{q}_i} = p_i$$

$$p_t = L + \dot{t} p_i (\frac{-\dot{q}_i}{\dot{t}^2}) = L - p_i \frac{dq_i}{dt} = -H$$

This already means that we have the constraint equation

$$\phi = p_t + H(q_i, p_i, t) = 0$$

The next surprise comes when we realise that the Hamiltonian vanishes

$$\tilde{H} = \dot{t}(-H) + \dot{t}\frac{dq_i}{dt} p_i - \dot{t} L = 0$$

However, this does not mean that the dynamics is frozen; for a physical observable $f(q_i, p_i, t)$, equation (1.1) reads

$$\dot{f} = \dot{t}\{f, \phi\}^\sim = \dot{t}\{f, p_t + H\}^\sim = \dot{t}\left(\frac{\partial f}{\partial t} + \{f, H\}^\sim\right)$$

$$= \dot{t}\left(\frac{\partial f}{\partial t} + \frac{\partial f}{\partial t}\frac{\partial H}{\partial p_t} + \frac{\partial f}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial f}{\partial p_t}\frac{\partial H}{\partial t} - \frac{\partial f}{\partial p_i}\frac{\partial H}{\partial q_i}\right) = \dot{t}\left(\frac{\partial f}{\partial t} + \{f, H\}\right)$$

Which is of course equivalent to the familiar equation

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \{f, H\}$$

1.5.2 Massless Systems

Let us consider a Lagrangian

$$L = \Pi_i(\underset{\sim}{q})\dot{q}^i - V(\underset{\sim}{q})$$

The equations of motion are

$$\frac{\partial \Pi_i}{\partial q^j}\dot{q}^j = \frac{\partial \Pi_j}{\partial q^i}\dot{q}^j - \frac{\partial V}{\partial q^i}$$

For now we assume that the matrix

$$\Omega_{ij} = (\frac{\partial \Pi_i}{\partial q_j} - \frac{\partial \Pi_j}{\partial q_i})$$

is not singular which means the system has proper, first order dynamics where all the velocities $\dot{q}_i$ are determined by the configuration coordinates $\{q_i\}$. Nevertheless let us continue with our constraint analysis algorithm; the constraints are

$$0 = F_i = -\Omega_{ij}\dot{q}^j - \frac{\partial V}{\partial q^i}$$

This is the end of the problem in the Lagrangian formalism; even if Ω_{ij} is singular, then there will be type-A constraints that stop the coordinates from being independent.

1.6 Exercises

1. Let the Lagrangian L depend on a parameter λ, show that

$$\frac{\partial H}{\partial \lambda} = -\frac{\partial L}{\partial \lambda}$$

2. For the Lagrangian $L = \frac{1}{2}q\dot{q}^2$ and initial values $q(0) = 1$, $\dot{q}(0) = -1$ find the solution $q(t)$ for all $t \in \mathbb{R}$.

3. In order to impose constraints

$$\gamma_i^I(\underset{\sim}{q})\dot{q}_i = 0$$

   on a Lagrangian system, one can add auxiliary coordinates Q_I and change the Lagrangian as

$$L\Big(\{\underset{\sim}{q}, \underset{\sim}{Q}\}, \{\underset{\sim}{\dot{q}}, \underset{\sim}{\dot{Q}}\}\Big) = L_0(\underset{\sim}{q}, \underset{\sim}{\dot{q}}) + Q_I\gamma_i^I(\underset{\sim}{q})\dot{q}_i$$

   Solve this Lagrangian and see if it actually works as intended.

4. Discuss the corrections necessary to account for explicit time dependencies in constrained Lagrangians and Hamiltonians.

5. For the Lagrangian

$$L = \dot{q}^i\Pi_i(\underset{\sim}{q}) - V(\underset{\sim}{q})$$

a)Show that the action is invariant under

$$\Pi_i \to \Pi_i + \partial_i F(\underset{\sim}{q})$$

also show that this transformation can be used to make the matrix $\partial_i \Pi_j$ anti-symmetric.

b) Assuming $\Omega_{ij} \equiv \partial_{[i} \Pi_{j]}$ is anti-symmetric and non-singular, use Darboux's theorem (See chapter C) to show that the Lagrangian can be re-written as

$$L = \frac{1}{2}(\dot{X}^i Y_i - X^i \dot{Y}_i) - H(X, Y)$$

c) Write down the Dirac formalism for this constrained Lagrangian and show that they mimic the phase space equations for the first order Hamiltonian dynamics where the Y_i are the momenta and the X^i are the coordinates:

$$\dot{Y} = -\frac{\partial H}{\partial X} \quad ; \quad \dot{X} = \frac{\partial H}{\partial Y}$$

Chapter 2

Classical Tensor Field Theory

By a "classical field theory", we mean a mechanical system with uncountably many degrees of freedom. Therefore, our default letter for degrees of freedom changes from q_i to $\phi_a(x)$ where x is some coordinate system on the index manifold and a runs over a finite (or at least countable) set. This is not relativistic yet; time is only a parameter and space is yet to be born. To make things relativistically covariant, we proceed to make the following changes:

- The index manifold is identified with *some* arbitrary spacelike manifold, Σ, embedded in the space-time.
- The Heisenberg picture is (inevitably) used to make sure that Σ is arbitrary and the fields ϕ_a are defined over the whole space-time.
- The integral over time in $S = \int L\, dt$ is replaced with the covariant integral over space-time: $S = \int dx\, \sqrt{-g}\mathcal{L}$.
- The Lagrangian density $\mathcal{L}$, being a function of the fields and their time derivatives, now *must* depend on the first order space-time derivatives in a covariant way: $\mathcal{L} = \mathcal{L}(\phi_a, \nabla_\mu \phi_a; x^\mu)$.
- The fields ϕ_a are covariant objects. In this chapter we focus on ϕ_a being tensors. It costs no generality to assume that all their indices are contravariant: $\phi_a^{\mu_1 \cdots \mu_{m_a}}$

Finally, we get

$$S = \int dx\, \sqrt{-g}\mathcal{L}(\phi_a, \nabla_\mu \phi_a; x^\alpha)$$

The variational principle leads to the Euler-Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial \phi_a} = \nabla_\mu \frac{\partial \mathcal{L}}{\partial \nabla_\mu \phi_a}$$

The action, S, (and therefore the equations of motion) are invariant under

$$\mathcal{L} \to \mathcal{L} + \varepsilon \nabla_\mu \mathcal{X}^\mu$$

for some $\mathcal{X}^\mu(\phi_a, \nabla_\mu \phi_a, x^\alpha)$. Therefore we expect an infinitesimal symmetry transformation to lead to such changes in the Lagrangian density. The transformation must look like

$$\phi_a \to \phi_a + \varepsilon \tau_a$$

The on-shell change in Lagrangian density is

$$\mathcal{L} \to \mathcal{L} + \varepsilon \nabla_\mu \frac{\partial \mathcal{L}}{\partial \nabla_\mu \phi_a} \tau_a$$

We immediately find out that corresponding to any symmetry transformation, there exists a conserved current

$$\nabla_\mu j^\mu = 0; \quad j^\mu \equiv \frac{\partial \mathcal{L}}{\partial \nabla_\mu \phi_a} \tau_a - \mathcal{X}^\mu$$

When discussing classical systems with finite or countable degrees of freedom, our first example of a symmetry was a time *shift* for time-independent Lagrangians. The corresponding symmetries in field theory are isometries; these are symmetries of the space-time that may be respected by the Lagrangian. Under such transformations, the fields are simply shifted along a symmetry direction.

In order to discuss the isometries of a given space-time manfiold, let us first introduce the geometric notion of Lie derivatives. Consider an infinitesimal coordinate transformation

$$x^\mu \to x^\mu - \varepsilon \xi^\mu.$$

Under such transformations, the components of a tensor $T^{\mu_1 \cdots \mu_m}_{\nu_1 \cdots \nu_n}$ change into

$$T^{\mu_1 \cdots \mu_m}_{\nu_1 \cdots \nu_n} \to T^{\mu_1 \cdots \mu_m}_{\nu_1 \cdots \nu_n} - \varepsilon \Big[\sum_{r=1}^{m} T^{\mu_1 \cdots \mu_{r-1} \alpha \mu_{r+1} \cdots \mu_m}_{\nu_1 \cdots \nu_n} \partial_\alpha \xi^{\mu_r} - \sum_{r=1}^{n} T^{\mu_1 \cdots \mu_m}_{\nu_1 \cdots \nu_{r-1} \alpha \nu_{r+1} \cdots \nu_n} \partial_{\nu_r} \xi^\alpha \Big]$$

On the other hand, the components of the same tensor, evaluated at a point with coordinates $x^\mu - \varepsilon \xi^\mu$ in the original coordinate system, are

$$T^{\mu_1 \cdots \mu_m}_{\nu_1 \cdots \nu_n} - \varepsilon \xi^\alpha \partial_\alpha T^{\mu_1 \cdots \mu_m}_{\nu_1 \cdots \nu_n}$$

The difference between these two expressions, divided by ε is defined to be the Lie derivative of the tensor field T along the vector field ξ^μ this is denoted by

$$\pounds_\xi T \equiv \xi^\alpha \partial_\alpha T^{\mu_1 \cdots \mu_m}_{\nu_1 \cdots \nu_n} - \sum_{r=1}^{m} T^{\mu_1 \cdots \mu_{r-1} \alpha \mu_{r+1} \cdots \mu_m}_{\nu_1 \cdots \nu_n} \partial_\alpha \xi^{\mu_r} + \sum_{r=1}^{n} T^{\mu_1 \cdots \mu_m}_{\nu_1 \cdots \nu_{r-1} \alpha \nu_{r+1} \cdots \nu_n} \partial_{\nu_r} \xi^\alpha$$

A Killing vector field, ξ^μ is one that satisfies $\pounds_\xi g = 0$ or written more explicitly

$$\nabla_{\{\alpha} \xi_{\beta\}} = 0$$

For such a field, we may write the second order derivatives in terms of the Riemann tensor

$$\begin{aligned}
\nabla_\alpha \nabla_\beta \xi_\mu &= -\nabla_\alpha \nabla_\mu \xi_\beta \\
&= -\nabla_\mu \nabla_\alpha \xi_\beta + [\nabla_\mu, \nabla_\alpha] \xi_\beta \\
&= \nabla_\mu \nabla_\beta \xi_\alpha + [\nabla_\mu, \nabla_\alpha] \xi_\beta \\
&= \nabla_\beta \nabla_\mu \xi_\alpha + [\nabla_\mu, \nabla_\beta] \xi_\alpha + [\nabla_\mu, \nabla_\alpha] \xi_\beta \\
&= -\nabla_\beta \nabla_\alpha \xi_\mu + [\nabla_\mu, \nabla_\beta] \xi_\alpha + [\nabla_\mu, \nabla_\alpha] \xi_\beta \\
&= -\nabla_\alpha \nabla_\beta \xi_\mu + [\nabla_\alpha, \nabla_\beta] \xi_\mu + [\nabla_\mu, \nabla_\beta] \xi_\alpha + [\nabla_\mu, \nabla_\alpha] \xi_\beta
\end{aligned}$$

Which may be rearranged to yield

$$\nabla_\alpha \nabla_\beta \xi_\mu = \frac{1}{2} \xi^\nu (R_{\alpha\beta\mu\nu} + R_{\mu\beta\alpha\nu} + R_{\mu\alpha\beta\nu}) = R_{\mu\beta\alpha\nu} \xi^\nu$$

Using this identity, we can prove the following nice result

$$[\nabla_\alpha, \pounds_\xi] T^{\mu_1 \cdots \mu_m}_{\nu_1 \cdots \nu_n} = \xi^\beta [\nabla_\alpha, \nabla_\beta] T^{\mu_1 \cdots \mu_m}_{\nu_1 \cdots \nu_n} - \sum_{r=1}^{m} T^{\mu_1 \cdots \mu_{r-1} \beta \mu_{r+1} \cdots \mu_m}_{\nu_1 \cdots \nu_n} \nabla_\alpha \nabla_\beta \xi^{\mu_r} + \sum_{r=1}^{n} T^{\mu_1 \cdots \mu_m}_{\nu_1 \cdots \nu_{r-1} \beta \nu_{r+1} \cdots \nu_n} \nabla_\alpha \nabla_{\nu_r} \xi^\beta = 0$$

where the last inequality is proved by substituting for the second order derivatives of ξ in terms of the Riemann tensor and cancelling the resulting terms with the anti-symmetric derivative operator acting on T.

2.1 The Energy-Momentum tensor(s)

Now that we are familiar with the Lie derivatives, let us consider the generalised shift transformation

$$\phi_a \to \phi_a + \varepsilon \pounds_\xi \phi_a$$

for some Killing vector ξ^μ. For a Lagrangian that only depends on the space-time position through the metric[1], we have

$$\mathcal{L} \to \mathcal{L} + \varepsilon\Big[\frac{\partial \mathcal{L}}{\partial \phi_a}\pounds_\xi \phi_a + \frac{\partial \mathcal{L}}{\partial \nabla_\alpha \phi_a}\nabla_\alpha \pounds_\xi \phi_a\Big]$$

$$= \mathcal{L} + \varepsilon\Big[\frac{\partial \mathcal{L}}{\partial \phi_a}\pounds_\xi \phi_a + \frac{\partial \mathcal{L}}{\partial \nabla_\alpha \phi_a}\pounds_\xi \nabla_\alpha \phi_a + \frac{\partial \mathcal{L}}{\partial g}\pounds_\xi g\Big]$$

$$= \mathcal{L} + \varepsilon \pounds_\xi \mathcal{L}$$

$$= \mathcal{L} + \varepsilon \xi^\mu \nabla_\mu \mathcal{L} = \mathcal{L} + \varepsilon \nabla_\mu \mathcal{L}\xi^\mu$$

and therefore we are dealing with a symmetry transformation. The conserved current will be

$$j^\mu = \mathcal{L}\xi^\mu - \frac{\partial \mathcal{L}}{\partial \nabla_\mu \phi_a}\pounds_\xi \phi_a = \mathtt{T}^{\mu\nu}\xi_\nu$$

where

$$\mathtt{T}^{\alpha\beta} \equiv T^{\alpha\beta}_{can.} + S^{\alpha\beta\gamma}\nabla_\gamma$$

with $T^{\alpha\beta}_{can.}$ as the canonical energy-momentum tensor

$$T^{\alpha\beta}_{can.} \equiv \mathcal{L}g^{\alpha\beta} - \frac{\partial \mathcal{L}}{\partial \nabla_\alpha \phi_a}\nabla^\beta \phi_a$$

and

$$S^{\alpha\beta\gamma} \equiv \frac{1}{2}\frac{\partial \mathcal{L}}{\partial \nabla_\alpha \phi_a}\sum_{r=1}^{m_a}\Big(\phi_a^{\mu_1\cdots\mu_{r-1}\gamma\mu_{r+1}\cdots\mu_{m_a}}g^{\mu_r\beta} - \phi_a^{\mu_1\cdots\mu_{r-1}\beta\mu_{r+1}\cdots\mu_{m_a}}g^{\mu_r\gamma}\Big)$$

so that $S^{\alpha\{\beta\gamma\}} = 0$; this symmetry property is natural since this tensor is born to be contracted with the covariant derivative of a Killing vector field, which renders its symmetric part irrelevant.

On shell, the canonical energy-momentum tensor satisfies

$$\nabla_\alpha T^{\alpha\beta}_{can.} = \nabla^\beta \mathcal{L} - \nabla_\alpha \frac{\partial \mathcal{L}}{\partial \nabla_\alpha \phi_a}\nabla^\beta \phi_a$$

$$= \nabla^\beta \mathcal{L} - \frac{\partial \mathcal{L}}{\partial g^{\mu\nu}}\nabla^\beta g^{\mu\nu} - \frac{\partial \mathcal{L}}{\partial \phi_a}\nabla^\beta \phi_a - \frac{\partial \mathcal{L}}{\partial \nabla_\alpha \phi_a}\nabla^\beta \nabla_\alpha \phi_a + \frac{\partial \mathcal{L}}{\partial \nabla_\alpha \phi_a}[\nabla^\beta, \nabla_\alpha]\phi_a$$

$$= S^{\alpha\mu\nu}R^\beta{}_{\alpha\mu\nu}$$

Using this we find that for any Killing vector ξ, the conservation equation gives

$$0 = \nabla_\alpha \mathtt{T}^{\alpha\beta}\xi_\beta = \frac{1}{2}\nabla_\alpha \xi_\beta\Big(T^{[\alpha\beta]}_{can.} - 2\nabla_\mu S^{\mu\alpha\beta}\Big)$$

Writing this on a flat (or symmetric enough) space-time, we find the identity

$$T^{[\alpha\beta]}_{can.} = 2\nabla_\mu S^{\mu\alpha\beta}$$

[1] This condition is sometimes called the minimal coupling condition and is justified by the equivalence principle.

which is an on-shell identity in every space-time.

Now define the Belinfante tensor as

$$B^{\mu\alpha\beta} \equiv S^{\mu\alpha\beta} + S^{\alpha\beta\mu} + S^{\beta\alpha\mu}$$

This definition is in fact the unique tensor built from $S^{\mu\alpha\beta}$ to satisfy

$$B^{\{\mu\alpha\}\beta} = 0; \quad B^{\mu[\alpha\beta]} = 2S^{\mu\alpha\beta}$$

for a general tensor $S^{\mu\alpha\beta}$ antisymmetric in its last two indices. The last property guarantees that the Belinfante energy-momentum tensor is symmetric.

$$T^{\alpha\beta}_{Bel.} \equiv T^{\alpha\beta}_{can.} - \nabla_\mu B^{\mu\alpha\beta}$$

This new tensor, is not only symmetric, but also conserved

$$\nabla_\alpha T^{\alpha\beta}_{Bel.} = \nabla_\alpha T^{\alpha\beta}_{can.} - \frac{1}{2}[\nabla_\alpha, \nabla_\mu]B^{\mu\alpha\beta}$$

$$= S^{\alpha\mu\nu} R^{\beta}{}_{\alpha\mu\nu} - \frac{1}{2}\big(- R_{\alpha\nu} B^{\nu\alpha\beta} + R_{\mu\nu} B^{\mu\nu\beta} + R_{\alpha\mu\beta\nu} B^{\mu\alpha\nu}\big)$$

$$= \frac{1}{4} R^{\beta}{}_{\alpha\mu\nu} S^{[\alpha\mu\nu]} = 0$$

These two properties then imply that for any Killing field ξ, the current $j'^\mu \equiv T^{\mu\nu}_{Bel.}\xi_\nu$ is conserved. In the exercises, the reader proves that this current is equivalent to j^μ in flat space-times and for translational isometries. In summary the Belinfante energy-momentum tensor is

- symmetric,
- conserved,
- equivalent to the canonical energy-momentum tensor in the sense that the total energy and momentum are the conserved charges corresponding to space-time translations; and
- allows one to write the conserved currents as $j^\mu = T^{\mu\nu}_{Bel.}\xi_\nu$.

2.2 The Hamiltonian Formalism

If we denote the momenta as

$$\pi_a(x) \equiv \frac{\partial \mathcal{L}}{\partial \nabla_0 \phi_a}$$

then the Hamiltonian is given by

$$H = \int_\Sigma d\mathbf{x}\, \sqrt{-g}\mathcal{H}; \quad \mathcal{H} \equiv \pi_a(x)\nabla_0\phi_a(x) - \mathcal{L}$$

This could not be more not covariant!

2.3 Exercises

1. Show that on a flat space time, the conserved quantities computed from the canonical and Belinfante tensors are the same.
$$\int d\mathbf{x}\, T^{0\mu}_{can.} = \int d\mathbf{x}\, T^{0\mu}_{Bel.}$$

2. ⚓ For the Lagrangian density
$$\mathcal{L} = -\frac{1}{2}\Phi^{\mu;\alpha}\Phi_{\mu;\alpha} - \frac{1}{2}m_\phi^2\Phi^\mu\Phi_\mu - \frac{1}{2}\Psi^{\mu\nu;\alpha}\Psi_{\mu\nu;\alpha} - \frac{1}{2}m_\psi^2\Psi^{\mu\nu}\Psi_{\mu\nu} + g\Psi^{\mu\nu;\alpha}\Phi_{\mu;\nu}\Phi_\alpha$$

   a) Write down the equations of motion.

   b) Compute the tensors $T^{\alpha\beta}_{can.}$, $S^{\mu\nu\alpha}$, $B^{\mu\nu\alpha}$, and $T^{\alpha\beta}_{Bel.}$.

Chapter 3

Classical Spinor Field Theory

When introducing field theories, we argued that the dynamical fields need to be covariant objects. We are already familiar with tensors and their transformation properties. In this section, we introduce another class of objects called spinors. We will also learn how to use spinor-valued tensors in order to get more complicated representations.

Let us start by finding the Lie algebra of the Lorentz group. An infinitesimal transformation keeps the Minkowski metric invariant

$$\big(\delta^\alpha_\mu + A^\alpha_\mu\big)\big(\delta^\beta_\nu + A^\beta_\nu\big)\eta^{\mu\nu} = \eta^{\alpha\beta}$$

which leads to the first order condition

$$A_{\{\alpha\beta\}} = 0$$

Such anti-symmetric tensors are written as linear combinations of the generators $J(\mu,\nu)$ where

$$J(\mu,\nu)^\alpha_\beta = \eta^{\mu\alpha}\delta^\nu_\beta - \delta^\mu_\beta\eta^{\nu\alpha}$$

The Lie algebra is given by

$$\big[J(\mu,\nu), J(\alpha,\beta)\big] = \eta^{\alpha\nu}J(\mu,\beta) - \eta^{\beta\nu}J(\mu,\alpha) - \eta^{\alpha\mu}J(\nu,\beta) + \eta^{\beta\mu}J(\nu,\alpha)$$

For our brand new representation, corresponding to spinors, the generators are given by

$$J_s(\mu,\nu) = \frac{1}{4}\big[\gamma(\mu),\gamma(\nu)\big]$$

where $\gamma(\mu)$ are a set of Dirac matrices satisfying the Clifford algebra

$$\big\{\gamma(\mu),\gamma(\nu)\big\} = 2\eta^{\mu\nu}$$

It is left as an exercise for the reader to show that the Clifford algebra implies the right Lie structure for J_s. A spinor is then defined as a mathematical object that transforms as

$$\psi \to \Big\{\exp\big[\sum_{\alpha<\beta}\omega_{\alpha\beta}J_s(\alpha,\beta)\big]\Big\}\psi$$

where the $\omega_{\alpha\beta}$ are the parameters for the Lorentz transformation. We will be using the short hand notations

$$\Lambda(\omega) \equiv \exp\big[\sum_{\alpha<\beta}\omega_{\alpha\beta}J(\alpha,\beta)\big]$$

$$\Lambda_{\frac{1}{2}}(\omega) \equiv \exp\big[\sum_{\alpha<\beta} \omega_{\alpha\beta} J_s(\alpha,\beta)\big]$$

Theorem 1. *For a one-form P_α, the expression*

$$\not{P} \equiv \sum_\mu \gamma(\mu) P_\mu$$

is a spinor operator; that is for a spinor ψ, the output $\not{P}\psi$ is also a spinor.

Proof. All we need is to check the transformation property of the expression, we have

$$\not{P}'\psi' = \sum_\mu \gamma(\mu)\Lambda^\nu_\mu(-\omega)P_\nu \Lambda_{\frac{1}{2}}(\omega)\psi$$

$$= \Lambda_{\frac{1}{2}}(\omega)\sum_\mu \Lambda_{\frac{1}{2}}(-\omega)\gamma(\mu)\Lambda_{\frac{1}{2}}(\omega)\Lambda^\nu_\mu(\omega)P_\nu\psi$$

To simplify this expression, we use the lemma

$$\Big[\gamma(\mu), J_s(\alpha,\beta)\Big] = \sum_\nu J(\alpha,\beta)^\mu_\nu \gamma(\nu)$$

which leads to

$$\not{P}'\psi' = \Lambda_{\frac{1}{2}}(\omega)\sum_{\mu,\rho} \Lambda^\mu_\rho(\omega)\gamma(\rho)\Lambda^\nu_\mu(-\omega)P_\nu\psi$$

$$= \Lambda_{\frac{1}{2}}(\omega)\not{P}\psi$$

□

It is also possible to construct Lorentz scalars, vectors, and tensors from spinors. One way to do so, is by forming the bilinear objects.

Theorem 2. *The array*

$$\Psi^{\mu_1\cdots\mu_n} \equiv \psi^\dagger M\gamma(\mu_1)\cdots\gamma(\mu_n)\psi$$

transforms as the components of a Lorentz tensor if and only if the matrix M satisfies

$$\big[\gamma^\dagger(\mu), \gamma^\dagger(\nu)\big]M = M\big[\gamma(\mu), \gamma(\nu)\big] \tag{3.1}$$

For such M, it is customary to denote the row-vector $\psi^\dagger M$ by $\bar{\psi}$.

3.1 Trace Technology From Clifford Algebra

The Clifford algebra alone, allows us to compute expressions of the form

$$\mathrm{Tr}\left[\gamma(\mu_1)\cdots\gamma(\mu_r)\right]$$

without using the explicit representation of the γ matrices. First, we need to define the matrix

$$\Gamma \equiv \frac{i}{(n+1)!}\sum_{\pi}\mathrm{sign}(\pi)\prod_{i=0}^{n}\gamma(\pi(0))\cdots\gamma(\pi(n)) = i\gamma(0)\cdots\gamma(n)$$

This satisfies

$$\Gamma^2 = -\gamma(0)\cdots\gamma(n)\gamma(0)\cdots\gamma(n) = (-1)^{\frac{n(n+1)}{2}}$$

$$\left\{\Gamma, \gamma(\mu)\right\} = \eta(\mu,\mu)(-1)^{\mu}\big(1+(-1)^{n}\big)\Gamma(\backslash\mu)$$

where

$$\Gamma(\backslash\mu) \equiv i\gamma(0)\cdots\gamma(\mu-1)\gamma(\mu+1)\cdots\gamma(n)$$

3.2 Dirac Spinors

For the $3+1$ dimensional space-time, Dirac introduced the 4×4 representation for spinors with Dirac matrices

$$\gamma(0) = \begin{pmatrix} i & & & \\ & i & & \\ & & -i & \\ & & & -i \end{pmatrix} \quad \gamma(1) = \begin{pmatrix} & & & i \\ & & i & \\ & -i & & \\ -i & & & \end{pmatrix}$$

$$\gamma(2) = \begin{pmatrix} & & & 1 \\ & & -1 & \\ & -1 & & \\ 1 & & & \end{pmatrix} \quad \gamma(3) = \begin{pmatrix} & & i & \\ & & & -i \\ -i & & & \\ & i & & \end{pmatrix}$$

For these matrices, the $\gamma(0)$ satisfies the condition (3.1). Dirac's Lagrangian read

$$\mathcal{L}_{\mathrm{Dirac}} \equiv \bar{\psi}(\not{\partial} - m)\psi$$

for which the Euler-Lagrange equations lead to the Dirac equation

$$(\not{\partial} - m)\psi = 0$$

3.3 Exercises

1. Prove theorem 2.

Part II

Prerequisites from Quantum Mechanics

Chapter 4

Canonical Quantisation and the Harmonic Oscillator

To quantise a mechanical system, one must find a complex Hilbert space $\mathscr{H}$ and Hermitian operators q_i, p_i that satisfy

$$[q_i, p_j] = i\delta_{ij}; \quad [q_i, q_j] = [p_i, p_j] = 0$$

This already means that the q_i have continuous spectra. It is then straightforward to show that the Hilbert space may be considered to be the linear space of all square-integrable functions $\psi(\underset{\sim}{q})$. The operators act as below

$$q_i\psi = q_i\psi(\underset{\sim}{q}); \quad p_i\psi = -i\partial_{q_i}\psi(\underset{\sim}{q})$$

The "state" of the system is nothing but a statistical description containing the expected values for all Hermitian operators. The expected values must be real, linear functions of the operators, therefore they are given by[1]

$$\langle f\rangle = \mathrm{Tr}\,\rho f$$

For some Hermitian density operator ρ. Normalization is necessary for consistency

$$1 = \langle 1\rangle = \mathrm{Tr}\,\rho$$

4.1 Quantum Dynamics

4.2 The Harmonic Oscillator

The (1D) Harmonic oscillator is the system described by the Hamiltonian

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2$$

[1] We will entirely limit our discussion to phase space functions that have unambiguous Hermitian expressions such as p^3, $\sin(aq + bp)$, etc. and not qp, p^q, etc.

The annihilation operator is

$$a \equiv \sqrt{\frac{\omega m}{2}} x + \frac{ip}{\sqrt{2\omega m}}$$

This satisfies

$$[a, a^\dagger] = 1$$

and allows us to write the Hamiltonian as

$$H = \omega(\frac{1}{2} + a^\dagger a)$$

The ground state is unique and found as the null vector of a:

$$\left(\sqrt{\frac{\omega m}{2}} x + \frac{1}{\sqrt{2\omega m}} \frac{d}{dx}\right) \psi_0(x) = 0$$

solved as

$$\psi_0(x) = (\frac{\omega m}{\pi})^{1/4} \exp\left(-\frac{\omega m x^2}{2}\right)$$

The nth mode is given by

$$\psi_n(x) = \frac{1}{\sqrt{n!}} a^{\dagger n} \psi_0(x) = \frac{1}{\sqrt{2^n n!}} (\frac{\omega m}{\pi})^{1/4} H_n(\sqrt{\omega m} x) \exp\left(-\frac{\omega m x^2}{2}\right)$$

where H_n is the nth Hermite polynomial. These polynomials can also be found via the recursive relation

$$H_{n+1} = 2xH_n - H_n'$$

with the initial condition $H_0(x) = 1$.

4.3 Mode Functions and The Bosonic Hilbert Space

Another, more abstract way of making the Hilbert space in which the conjugate operators q, p live is by postulating the existence of an operator a, called the annihilating operator satisfying

$$[a, a^\dagger] = 1$$

and a unique, normalised vector $|0\rangle$ satisfying

$$a|0\rangle = 0$$

The rest of the Hilbert space is then a linear combination of normalized states

$$|n\rangle \equiv \frac{a^{\dagger n}}{\sqrt{n!}} |0\rangle$$

The canonical variables can be described as a linear combination of the annihilation operator and its Hermitian conjugate, the creation operator

$$p = ua + u^* a^\dagger \quad ; \quad q = va + v^* a^\dagger$$

provided that

$$\text{Im}(u^* v) = \frac{1}{2}$$

4.3.1 An Application: Time Dependent Oscillator

The Hamiltonian of the time dependent oscillator is

$$H = \frac{p^2}{2m(t)} + \frac{1}{2}m(t)\omega^2(t)q^2$$

For these Hamiltonians, all the dynamics can be encoded in the time dependence of the mode functions $u(t)$, and $v(t)$ while keeping both the state $|\psi\rangle$, and the creation-annihilation operators, a, $a^\dagger$ constant. The first equation of motion is

$$\frac{dq}{dt} = i[H,q] = \frac{p}{m(t)} \quad \Longleftrightarrow \quad u = m\dot{v}$$

This makes the canonical commutation relation equivalent to

$$\mathrm{Im}(\dot{v}^* v) = \frac{1}{2m(t)}$$

The second equation of motion is

$$\frac{dp}{dt} = i[H,p] = -\omega^2(t)m(t)q$$

equiavelnt to

$$\ddot{v} + \frac{\dot{m}}{m}\dot{v} + \omega^2(t)v = 0$$

4.4 Coherent and Squeezed States

Let us start by considering a Hamiltonian

$$H = \sum_i \left(\frac{p_i^2}{2m_i} + \frac{1}{2}m_i\omega_i^2 x_i^2 \right)$$

4.5 Exercises

1. Let $\mathbf{r} = (q,p)$, for some vector $\mathbf{x}$, define the Weyl operator $D(\mathbf{x})$ as

$$D(\mathbf{x}) \equiv \exp\left(\mathbf{x}^T \sigma_2 \mathbf{r}\right)$$

a) Prove

$$[D(\mathbf{x}), \mathbf{r}] = -\mathbf{x}D(\mathbf{x})$$

b) Show that for all vectors $\mathbf{x}$ and $\mathbf{y}$, we have

$$D(\mathbf{x}+\mathbf{y}) = D(\mathbf{x})D(\mathbf{y})\exp\left(\mathbf{x}^T \sigma_2 \mathbf{y}\right)$$

c) Prove the following identity

$$D(-\mathbf{x})\mathbf{r}D(\mathbf{x}) = \mathbf{r} + \mathbf{x}$$

d) We may encode the real vector $\mathbf{x} = (x_q, x_p)$ into a complex variable z as

$$z \equiv \sqrt{\frac{\omega m}{2}} x_q + \frac{i}{\sqrt{2\omega m}} x_p$$

Show that

$$D(\mathbf{x}) = \exp\left(z a^\dagger - z^* a\right) \equiv D_c(z)$$

where a is the annihilation operator for the harmonic operator with parameters (m, ω).

e) Show that

$$D_c(z) = \exp\left(-\frac{1}{2}|z|^2\right) \exp\left(z a^\dagger\right) \exp(-z^* a)$$

f) Show that

$$\langle n+s| D_c(z) |n\rangle = e^{-|z|^2/2} z^s \sqrt{n!(n+s!)} \sum_{l=0}^{n} \frac{(-|z|^2)^l}{l!(l+s)!(n-l)!}$$

g) Show that

$$\int d^2 z \, D_c(z) |0\rangle \langle 0| D_c(-z) = \pi$$

h) For a complex number $\xi = \rho e^{i\varphi}$ define the unitary squeezing operator

$$S(\xi) = \exp\left(\xi^* a^2 - \xi a^{\dagger 2}\right)$$

Show that

$$S(-\xi) a S(\xi) = \cosh(2\rho) a + \sinh(2\rho) e^{i\varphi} a^\dagger$$

i) Find the covariance matrix

$$\langle 0| S(-\xi) r_i r_j S(\xi) |0\rangle$$

2. Show that the ladder states $|n\rangle$ form a basis for the Hilbert space.

3. Consider the time independent harmonic oscillator where $m(t) = m$ and $\omega(t) = \omega$ are time independent.

a) Show that a unique (up to a phase) choice of $v(t)$ exists that makes the Hamiltonian take the form

$$H = \omega(a^\dagger a + \frac{1}{2})$$

b) Then consider a more generic mode function solution of the form

$$v'(t) = \alpha v(t) + \beta v^*(t) + \gamma \quad ; \quad u' = m\dot{v}' + \delta$$

Find conditions on the parameters $\alpha, \beta, \gamma, \delta$ that make this a solution. These are called squeezed, coherent states.

c) Find the first two moments for the ground state

$$\langle x, p, x^2, p^2 \rangle$$

as functions of time. Show that only solutions from part a have time independent averages.

4. **(Second Canonical Quantization)** Start from the ”massless” Lagrangian

$$L = \frac{1}{2}(\dot{x}y - x\dot{y}) - H(x, y)$$

a) Show that the classical physical phase space is the 2 dimensional submanifold of the 4 dimensional formal phase space determined by constraint equations

$$p_x - \frac{1}{2}y = p_y + \frac{1}{2}x = 0$$

b) Show that there is no subspace of the Hilbert space in which both constraints are exactly satisfied.

Instead of solving the constraint equations, let us focus on the 'meta'-Hamiltonian

$$H_{\text{meta}} \equiv \frac{1}{2}\left[(p_x - \frac{1}{2}y)^2 + (p_y + \frac{1}{2}x)^2\right]$$

and see if its ground state can correspond to anything physical.

c) Write down the classical equations of motion for (x, y) evolving with H_{meta} and show that they mimic the motion of a charged particle in a plane with constant normal magnetic field.

d) Define

$$X \equiv x - 2p_y \quad ; \quad Y \equiv y + 2p_x$$

show that

$$[X, H] = [Y, H] = 0 \quad ; \quad [X, Y] = 4i$$

make sense of the factor of 4. (Hint: what are the values of X, Y when evaluated at the physical submanifold?)

e) Define

$$a \equiv \frac{1}{\sqrt{2}}\left((p_x - \frac{y}{2}) - i(p_y + \frac{x}{2})\right) \quad ; \quad b \equiv \frac{1}{\sqrt{2}}\left((p_x + \frac{y}{2}) + i(p_y - \frac{x}{2})\right)$$

show that

$$[a, b] = [a, b^\dagger] = 0 \quad ; \quad [a, a^\dagger] = [b, b^\dagger] = 1$$

$$H_{\mathrm{meta}} = a^\dagger a + \frac{1}{2}$$

f) Show that the ground states are of the form

$$\psi(x, y) = f(x - iy) \times \exp\left(-\frac{x^2 + y^2}{4}\right)$$

with $f(z)$ being any analytic function. Hint: prove

$$a = -i\sqrt{2}(\bar{\partial} + \frac{z}{4}) \quad ; \quad a^\dagger = -i\sqrt{2}(\partial - \frac{\bar{z}}{4}) \quad ; \quad b = -i\sqrt{2}(\partial + \frac{\bar{z}}{4}) \quad ; \quad b^\dagger = -i\sqrt{2}(\bar{\partial} - \frac{z}{4})$$

The physical subspace may not be exactly achievable in the quantum world, but the ”band” with $n_a = 0$ is the next best thing.

g) Write down x and y in terms of the creation and annihilation operators and show that projected on a specific band with $n_a = \nu$, we get back the physical commutation relationship

$$[x_\nu, y_\nu] = i$$

Chapter 5

Perturbation Theory

5.1 Time Dependent Schroedinger Equation

The time dependent Schroedinger equation is

$$i\frac{d}{dt}\left|\psi;t\right\rangle = H(t)\left|\psi;t\right\rangle$$

And the solutions are encoded in the Schroedinger propagator $U(t)$

$$\left|\psi;t\right\rangle = U(t)\left|\psi;0\right\rangle .$$

The propagator satisfies

$$i\frac{d}{dt}U(t) = H(t)U(t); \quad U(0) = \mathbb{I}$$

this implies

$$U(t) = \mathbb{I} - i\int_0^t dt'\, H(t')U(t')$$

The integral equation above, suggests the so called Dyson series as the Schroedinger propagator

$$U(t) = \mathbb{I} + \sum_{n=1}^{\infty}(-i)^n \int_0^t dt_1 \int_0^{t_1} dt_2 \,\cdots \int_0^{t_{n-1}} dt_n\, H(t_1)\cdots H(t_n)$$

This may also be written in the following, compact form

$$U(t) = \mathscr{T}\Big\{\exp\Big[-i\int_0^t dt'\, H(t')\Big]\Big\}$$

where the symbol $\mathscr{T}$ denotes the time ordering: in a product of operators from different times, the order of multiplication from left to right is chronologically decreasing.

If $U_N(t)$ denotes the truncated series after N terms, it is easy to show that

$$\big|\big|U(t) - U_N(t)\big|\big|_2 \leq \frac{t^{n+1}}{(n+1)!}\Big(\sup_{\tau\in[0,t]}\big|\big|H(t)\big|\big|_2\Big)^{n+1}$$

However, for the case of field theories, were $||H(t)||_2 = \infty$ at all times, this inequality is of no practical use; there we only *hope* that the Dyson series yields proper, relevant results.

5.2 The Interaction Picture

Now consider the case of a quantum mechanical system with a Hamiltonian in the form

$$H = H_0 + H_1$$

Assume that we know the *unperturbed* propagator $U_0(t)$ satisfying

$$i\frac{d}{dt}U_0(t) = H_0(t)U_0(t)$$

and want to write the full propagator as

$$U(t) = U_0(t)V(t)$$

The relevant differential equation is

$$\frac{d}{dt}V(t) = -i\big[U_0^\dagger(t)H_1U_0(t)\big]V(t) = -iH_1^0(t)V(t)$$

In words, the compensation propagator, $V(t)$, satisfies the Schroedinger equation with the unperturbed Heisenberg version of the perturbation Hamiltonian. In general, this is a time dependent Hamiltonian; however, since the perturbation term H_1 is asuumed to be small, we may hope to get proper results using the Dyson series.

5.3 The Adiabatic Theorem

Consider a family of Hamiltonians $H(\lambda^a)$. For now assume that for each λ, the Hamiltonian is non-degenerate.

$$H(\lambda^a)\,|n,\lambda^a\rangle = E_n(\lambda^a)\,|n,\lambda^a\rangle$$

Now for a smooth path $\lambda^a(\sigma)$ for $\sigma \in [0,1]$ consider the time dependent Hamiltonian

$$H(t) = H\big(\lambda^a(t/T)\big); \quad t \in [0,T]$$

The adiabatic theorem is stated as follows

$$\lim_{T\to\infty} U(T)\,|n,\lambda^a(0)\rangle\,\langle n,\lambda^a(0)|\,U^\dagger(T) = |n,\lambda^a(1)\rangle\,\langle n,\lambda^a(1)|$$

To prove this, let us start by writing

$$c_{mn}(t) \equiv \exp\left[i\int_0^t dt'\,E_m\big(\lambda^a(t'/T)\big)\right]\langle m,\lambda^a(t/T)|\,U(t)\,|n,\lambda^a(0)\rangle$$

Differentiating with respect to $\sigma = t/T$ we get

$$\frac{dc_{mn}}{d\sigma} = \sum_l \exp\left[iT\int_0^\sigma d\sigma'\,\big[E_m\big(\lambda^a(\sigma')\big) - E_l\big(\lambda^a(\sigma')\big)\big]\right]c_{ln}(\sigma)\Big(\frac{d\lambda^a}{d\sigma}\frac{\partial}{\partial\lambda^a}\langle m,\lambda^a(t)|\Big)\,|l,\lambda^a(\sigma)\rangle$$

Now in the $T \to \infty$ limit, all the $l \neq m$ terms will be irrelevant[1] and therefore we get to re-write the evolution as

$$c_{mn}(\sigma) = c_{mn}(0) \exp\Big\{ \int_{\lambda(0)}^{\lambda(\sigma)} d\lambda^a \,\big(\frac{\partial}{\partial \lambda^a} \langle m, \lambda^a | \big) \, |m, \lambda^a\rangle \Big\}$$

This completes the proof for the adiabatic theorem as soon as we substitute $c_{mn}(0) = \delta_{mn}$.

The $non - dynamical$ phase that the amplitude c_{mn} takes during the adiabatic evolution is called the Berry phase. In fact if we define the following (real) 1-form

$$B_{m,a} \equiv i \langle m, \lambda^a | \frac{\partial}{\partial \lambda^a} |m, \lambda^a\rangle$$

we get to write the Berry phase as

$$\gamma_m = \int d\lambda^a \, B_{m,a}$$

5.4 The Correlation Functions

Suppose we want to compute the correlation function

$$\langle O_1(t_1) \cdots O_n(t_n) \rangle_{|E\rangle\langle E|}$$

for some eigenstate of the perturbed Hamiltonian

$$H |E\rangle = (H_0 + H_1) |E\rangle = E |E\rangle$$

This would be the same as

$$\langle E| e^{iH(t_1 - t_0)} O_1(t_0) e^{iH(t_2 - t_1) O_2(t_0)} \cdots e^{iH(t_n - t_{n-1})} O_n(t_0) e^{iH(t_0 - t_n)} |E\rangle$$

Ideally, we would want to deal with the unperturbed state $|E_0\rangle$ instead of the perturbed state $|E\rangle$. To do this, we first introduce the adiabatic function $\sigma(t)$.

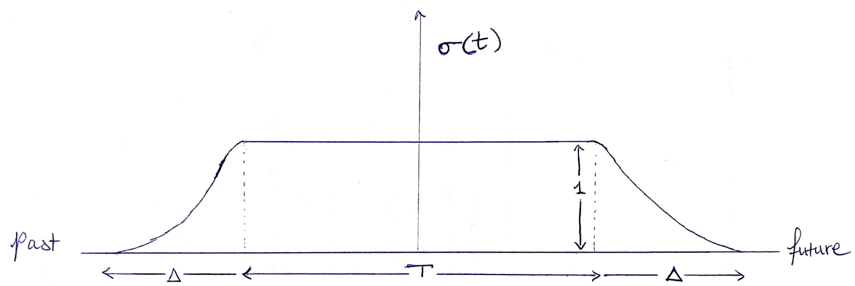
[1]To see this, consider some $\delta\sigma$ such that

$$\frac{1}{|T(E_m - E_l)|} \ll \delta\sigma \ll 1$$

and show that the change

$$\frac{\partial}{\partial c_{ln}} \frac{\delta c_{mn}}{\delta \sigma} \ll 1$$

for $m \neq l$



The adiabatic function $\sigma(t)$. We are dealing with the limit $T, \Delta, T/\Delta \to \infty$

Replacing the perturbed, time independent Hamiltonian with the following

$$H(t) = H_0 + \sigma(t) H_1$$

we see that in the far past and future times, the perturbation is turned off but for all physical, finite times, the Hamiltonian remains the same.

In this section it will be more convenient to work with the double-argument compensator operator

$$V(t_1, t_0) \equiv \mathscr{T}\Big\{ \exp\Big[-i \int_{t_0}^{t_1} H_1^0(t)\sigma(t)\, dt \Big] \Big\}$$

Now, using the adiabatic theorem, we know that the state $|E\rangle$ corresponds to some unperturbed state $|E_0\rangle$ in the far past times. Therefore, the correlation function becomes

$$\langle O_1(t_1) \cdots O_n(t_n) \rangle_{|E\rangle\langle E|} = \langle E_0| V(\text{past}, t_1) O_1^0(t_1) V(t_1, t_2) O_2^0(t_2) \cdots O_n^0(t_n) V(t_n, \text{past}) |E_0\rangle$$

The adiabatic theorem also assures us that in the adiabatic limit

$$V(\text{future}, \text{past}) |E_0\rangle = e^{i\alpha} |E_0\rangle$$

and therefore

$$\langle O_1(t_1) \cdots O_n(t_n) \rangle_{|E\rangle\langle E|} = \frac{\langle E_0| V(\text{past}, t_1) O_1^0(t_1) V(t_1, t_2) O_2^0(t_2) \cdots O_n^0(t_n) V(t_n, \text{future}) |E_0\rangle}{\langle E_0| V(\text{future}, \text{past}) |E_0\rangle}$$

The operator $S \equiv V(\text{future}, \text{past})$ is called the Scattering matrix or simply the S-matrix for reasons that become clear later on. When computing the S-matrix, we may forget about the small proportions of time duration when the perturbation Hamiltonian is not fully turned on since their contribution to each term in the Dyson formula is infinitesimal.

If, without loss of generality, we assume that $t_1 > t_2 > \cdots > t_n$ are time ordered, then the following formula follows as the final result for this section.

$$\boxed{\langle O_1(t_1) \cdots O_n(t_n) \rangle_{|E\rangle\langle E|} = \frac{\Big\langle E_0 \Big| \mathscr{T}\{O_1^0(t_1) \cdots O_n^0(t_n) S\} \Big| E_0 \Big\rangle}{\langle E_0| S |E_0\rangle}}$$

5.5 Exercises

1. For the perturbed harmonic oscillator

$$H_0 = \frac{p^2}{2} + \frac{x^2}{2}; \quad H_1 = -\lambda x$$

show that the perturbed ground state is centered at

$$\langle x \rangle_g = \lambda$$

Chapter 6

Scattering Theory

In this chapter, we develop the theoretical tools that describe scattering amplitudes, cross sections, and other practical quantities concerning scattering experiments. The Hilbert space corresponds to that of a non relativistic particle living on a Galilean space-time; namely the space of square-integrable wave functions on $\mathbb{E}^n$.

6.1 Gaussian Wave Packets

In this section, we focus on *single-particle* states that best describe a freely moving particle; i.e. Gaussian wave packets. In Fourier space

$$\tilde{\psi}(\mathbf{k}) = A \exp\left[-(\mathbf{k}-\bar{\mathbf{k}})^T \Sigma (\mathbf{k}-\bar{\mathbf{k}}) - i\bar{\mathbf{x}}^T \mathbf{k}\right]$$

We will generally ignore the normalisation constant A in this chapter. The three parameters $(\bar{\mathbf{x}}, \bar{\mathbf{k}}, \Sigma) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{C}^{n\times n}$ are defined in a way so that

$$\langle x_i \rangle = \bar{x}_i, \quad \langle p_i \rangle = \bar{k}_i, \quad \mathrm{Cov}(p_i, p_j) = \frac{1}{4}\big(\mathrm{Re}[\Sigma]\big)^{-1}_{ij}, \quad \mathrm{Cov}(x_i, x_j) = \Big(\mathrm{Re}\left[\Sigma^{-1}\right]\Big)^{-1}_{ij}$$

Under the free Hamiltonian $H = p^2/2m$, the wave function remains Gaussian and the parameters evolve as

$$\Sigma(t) = \Sigma(0) + \frac{it}{2m}\mathbb{I}, \quad \bar{\mathbf{k}}(t) = \bar{\mathbf{k}}(0), \quad \bar{\mathbf{x}}(t) = \bar{\mathbf{x}}(0) + \frac{t\bar{\mathbf{k}}}{m}$$

6.2 Spherical Schroedinger Equation

The time-independent Schroedinger equation is

$$\Big(-\frac{\nabla^2}{2m} + V(\mathbf{x})\Big)\psi(\mathbf{x}) = E\psi(\mathbf{x})$$

For a spherically symmetric potential, it is best to work in the spherical coordinates (x, Ω), then the equation is solved as

$$\psi(\mathbf{x}) = Y_{lm}(\Omega)\frac{u(x)}{x^{(n-1)/2}}$$

where

$$(\nabla_\Omega^2 + \lambda_l^2) Y_{lm}(\Omega) = 0; \quad \int d\Omega \, Y^*_{lm}(\Omega) Y_{l'm'}(\Omega) = \delta_{ll'}\delta_{mm'}$$

and

$$\frac{-1}{2m}\frac{d^2u}{dx^2} + \Big[V(x) + \frac{\lambda_l^2 + (n-1)(n-3)/4}{2mx^2}\Big]u(x) = Eu(x); \quad u(0) = 0; \quad \int |u(x)|^2 dx = 1$$

The main purpose of introducing the spherical equations is to write down a plane wave $\psi = e^{ikx\cos\theta}$ - where θ is the polar angle i.e. $x^1 = x\cos\theta$ -in terms of the spherical *partial waves*. Here the relevant $Y_{lm}(\Omega)$ are the generalised Legendre polynomials $Y_{lm} \to Q_{\ell n}(\cos\theta)$ (remember n is the dimension of the space). Here $Q_{\ell n}$ are determined via the equations

$$(1-y^2)\frac{d^2Q}{dy^2} - (n-1)y\frac{dQ}{dy} + \Lambda_{\ell n}^2 Q = 0$$

$$\int_{-1}^{+1} dy \, w_n(y) Q_{\ell n}(y) Q_{\ell' n}(y) = \delta_{\ell\ell'}$$

$$w_n(y) = \frac{2\pi^{(n-1)/2}}{\Gamma\big(\frac{n-1}{2}\big)}(1-y^2)^{(n-3)/2}$$

The first two polynomials are easy to compute

$$Q_{0n}(x) = \sqrt{\frac{\Gamma(n/2)}{2\pi^{n/2}}}; \quad \Lambda_{0n} = 0$$

$$Q_{1n}(x) = \sqrt{\frac{\Gamma(n/2+1)}{\pi^{n/2}}}x; \quad \Lambda_1 = \sqrt{n-1}$$

$$\vdots$$

In general it is possible to show that $\Lambda_{\ell n} = \ell(\ell + n - 2)$ (Cf. Exercises).

Since the plane wave does not blow up in the origin, we have the following expansion

$$e^{ikx\cos\theta} = \sum_{\ell=0}^{\infty} A_\ell \frac{Q_{\ell n}(\cos\theta)}{(kx)^{n/2-1}} J_{\ell+\frac{n}{2}-1}(kx)$$

6.3 Stationary Unbounded States

Now let us add a localized potential term to our Hamiltonian

$$H = \frac{p^2}{2m} + V(x), \quad \lim_{||\mathbf{x}|| \to \infty} V(\mathbf{x}) = 0$$

Apart from the $E < 0$ part of its spectrum, where the wave functions decay exponentially for large $\mathbf{x}$, we may also look for stationary states with $E > 0$. These states will not be properly normalisable, however a superposition of them may be normalised properly. For a wave vector $\mathbf{k}$, define

$$\psi_{\mathbf{k}}(\mathbf{x}) = e^{i\mathbf{k}.\mathbf{x}} + f(\Omega)\frac{e^{ikx}}{x^{(n-1)/2}}\big(1 + o(1)\big)$$

where the decaying $o(1)$ term denotes a decaying function of r.

As proved in the exercises, the probability current vector

$$\mathbf{J} \equiv \frac{1}{m} \operatorname{Im}\left[\psi^* \nabla \psi\right]$$

satisfies the continuity equation.

$$\frac{\partial |\psi|^2}{\partial t} + \nabla.\mathbf{J} = 0$$

An evaluation of this current at large r yields a connection between $f(\Omega)$ and the differential cross section as

$$\boxed{D(\Omega) = |f(\Omega)|^2}$$

6.4 Perturbative Ladder

The Schrödinger equation can be re-written as

$$(\nabla^2 + k^2)\psi = 2mV\psi$$

In the absence of the potential, we decided that the incident wave is

$$\psi_0(\mathbf{x}) = \exp(i\mathbf{k}.\mathbf{x})$$

In the presence of the potential, we may expand the wavefunction in terms of the potential as

$$\begin{aligned}\psi &= \psi_0 + (\nabla^2 + k^2)^{-1}(2mV)\psi_0 + (\nabla^2 + k^2)^{-1}(2mV)(\nabla^2 + k^2)^{-1}(2mV)\psi_0 + \cdots \\ &= \left[1 - (\nabla^2 + k^2)^{-1} \times (2mV)\right]^{-1} \psi_0\end{aligned}$$

Let's now focus on the one key part in this expansion, that is $(\nabla^2 + k^2)^{-1}$. We are looking for the Green function that solves

$$(\nabla^2 + k^2)\psi = \delta(\mathbf{x})$$

Imposing the boundary condition that rules out incoming waves, we get the solution

$$\psi = A(kx)^{1-n/2} H_{\frac{n}{2}-1}(kx)$$

where the Hankel function H_α is

$$H_\alpha(x) = \frac{J_{-\alpha}(x) - e^{-i\alpha\pi} J_\alpha(x)}{i \sin(\alpha\pi)}$$

The constant A is set such that for $kx \ll 1$, we get

$$\psi \sim -\frac{\Gamma(n/2-1)}{4\pi^{n/2} x^{n-2}}$$

6.5 Exercises

1. Starting from the separation of variables

$$\psi(\mathbf{x}) = Y_{\lambda\mu}(\Omega) \frac{u(x)}{x^{(n-1)/2}}$$

show that the radial Schroedinger equation is equivalent to a 1D problem with

$$V_{eff}(x) = V(x) + \frac{\lambda^2 + (n-1)(n-3)/4}{2mx^2}$$

2. Regarding the n dimensional Legendre differential equation

$$(1-y^2)\frac{d^2Q}{dy^2}-(n-1)y\frac{dQ}{dy}+\Lambda^2 y=0$$

Let

$$Q(y)=\sum_{n=0}^{\infty}q_n y^n$$

and find a recursive equation for q_{n+2} in terms of q_n. Show that Q is singular near $\cos\theta=1$ unless the series is cut off somewhere with $q_{\ell+2}=0$. From there, deduce the quantization rule $\Lambda^2_{\ell n}=\ell(\ell+n-2)$

Chapter 7

Path Integral Quantisation

In this chapter, we limit ourselves to dynamical systems with a Lagrangian in the form

$$L = \frac{1}{2}\dot{q}_i\dot{q}_j M_{ij}(t) - V(t) + A_i(t)\dot{q}_i$$

and introduce an equivalent quantisation formalism to find the expected values of different observables. This approach will specifically be useful for proving certain theorems in quantum field theory.

7.1 The Fundamental Formula

We know that the canonical quantisation of the Lagrangian

$$L = \frac{1}{2}\dot{q}_i\dot{q}_j M_{ij}(t) - V(t) + A_i(t)\dot{q}_i$$

leads to Schroedinger's wave equation

$$i\frac{\partial}{\partial t}\psi(q,t) = H\psi(q,t)$$

with

$$H = -\frac{1}{2}M^{-1}_{ij}\frac{\partial}{\partial q_i}\frac{\partial}{\partial q_j} + iM^{-1}_{ij}A_i\frac{\partial}{\partial q_j} + V.$$

However, the same dynamical equation may be derived using the path integral formalism. In this formalism, the probability amplitude for a transition from q_1 at some time t_1 to q_2 at t_2 is found by *summing* over all *paths* $q_i(t)$ that have proper starting and end points.

$$\text{Amp.}\Big[(q_1,t_1) \longrightarrow (q_2,t_2)\Big] \propto \sum_{\text{paths}} e^{iS[\text{path}]}$$

The constant of proportionality depends on the resolution of the set of paths we are summing over. To fix this constant, let us focus on a very short time interval $[t, t+\delta t]$. In this regime, we may approximate any path with a straight line and write

$$\psi(q,t+\delta t) \propto \int dr\,\psi(q-r,t)\exp\Big[i\Big(r_i r_j\frac{M_{ij}}{2\delta t} + r_i A_i - \delta t V\Big)\Big]$$

$$= \int dr\,\psi(q-r,t)\exp\Big[i\Big((r_i + \delta t M^{-1}_{ik}A_k)(r_j + \delta t M^{-1}_{jl}A_l)\frac{M_{ij}}{2\delta t} - \delta t\big(V + 1/2A_iA_jM^{-1}_{ij}\big)\Big)\Big]$$

$$\propto \psi(q,t) - i\delta t\Big[\big(V + \frac{1}{2}A_i A_j M^{-1}_{ij}\big) + iM^{-1}_{ij}A_i\frac{\partial\psi(q,t)}{\partial q_j} - \frac{1}{2}M^{-1}_{ij}\frac{\partial^2\psi(q,t)}{\partial q_i\partial q_j}\Big] + o(\delta t)$$

This shows that the only consistent dynamics that we may get from the path integral formalism, is that of Schoroedinger's equation.[1] We invent a special notation for this specific choice of the constant of proportionality and write

$$\boxed{\text{Amp.}\Big[(q_1,t_1) \longrightarrow (q_2,t_2)\Big] = \int_{(q_1,t_1)}^{(q_2,t_2)} Dq\, e^{iS[q]}}$$

Comparing this with the canonical quantisation formalism we get

$$\Big\langle q_2\Big|\mathscr{T}\Big\{\exp\Big[-i\int_{t_1}^{t_2} dt\, H(t)\Big]\Big\}\Big|q_1\Big\rangle = \int_{(q_1,t_1)}^{(q_2,t_2)} Dq\, e^{iS[q]}$$

In general, we may define

$$\boxed{\Big\langle \psi_2\Big|\mathscr{T}\Big\{\exp\Big[-i\int_{t_1}^{t_2} dt\, H(t)\Big]\Big\}\Big|\psi_1\Big\rangle = \int_{(\psi_1,t_1)}^{(\psi_2,t_2)} Dq\, e^{iS[q]} = \int dq_1 dq_2\, \psi_2^*(q_2)\psi_1(q_1)\int_{(q_1,t_1)}^{(q_2,t_2)} Dq\, e^{iS[q]}}$$

7.2 The Euclidean Time

For time independent Hamiltonians, it is more often than not that we are interested in thermal states

$$\rho(\beta) \equiv \frac{e^{-\beta H}}{\mathrm{Tr}\, e^{-\beta H}}$$

Specially the ground state corresponding to $\beta = \infty$ is of great interest in quantum field theories. The operator $e^{-\beta H}$ satisfies the Euclidean Schroedinger equation

$$\frac{d}{d\beta}e^{-\beta H} = -He^{-\beta H}$$

This is similar to Schroedinger's equation if we define the Euclidean time as $t \equiv -i\beta$. In terms of β, the action becomes

$$idS = -\frac{1}{2}M_{ij}(-i\beta)\frac{dq_i}{d\beta}\frac{dq_j}{d\beta} - V(-i\beta)d\beta + iA_i(-i\beta)dq_i$$

here, the expressions depending on $-i\beta$ should be interpreted as the analytic extension of these functions from the real axis to the imaginary axis. From now on, we focus on the special case where the Hamiltonian is time-independent and therefore drop the time/temperature arguments.

7.3 Ground State Expectation Values

The path integral formalism may not be considered as a complete quantisation formalism unless we provide a path integral formula for correlation functions. In general we are interested in

$$\langle\Omega|\,\mathscr{T}\big\{q_{i_1}(t_1)\cdots q_{i_n}(t_n)\big\}\,|\Omega\rangle$$

[1]There is a slight, irrelevant difference here. The potential $V(q)$ now has an extra term that is only a function of time. This won't affect any observation and we need not worry about it.

It is easy to show that the following recipe is consistent with the canonical formalism

$$\boxed{\langle\Omega|\,\mathscr{T}\big\{q_{i_1}(t_1)\cdots q_{i_n}(t_n)\big\}\,|\Omega\rangle = \frac{\int_{\Omega,\text{past}}^{\Omega,\text{future}} Dq\, e^{iS[q]} q_{i_1}(t_1)\cdots q_{i_n}(t_n)}{\int_{\Omega,\text{past}}^{\Omega,\text{future}} Dq\, e^{iS[q]}}}$$

An advantage of the Euclidean time is that we don't need to specify the initial and final states. From now on, any path integral without limits must be interpreted as a Euclidean time integral over the whole imaginary axis. Moreover, any expectation value $\langle O\rangle$ with no specific states, should be interpreted as the ground state expectation value. Therefore we are allowed to write

$$\boxed{\langle\mathscr{T}\big\{q_{i_1}(-i\beta_1)\cdots q_{i_n}(-i\beta_n)\big\}\rangle = \frac{\int Dq\, e^{iS[q]} q_{i_1}(-i\beta_1)\cdots q_{i_n}(-i\beta_n)}{\int Dq\, e^{iS[q]}}}$$

7.4 Schwinger-Dyson equation and Ward Identities

In classical systems the symmetry of the product rule ($AB = BA$) allows us to freely move differentiation operators inside correlation functions; for example

$$\frac{d}{dt_1}\langle q_{i_1}(t_1)q_{i_2}(t_2)\rangle = \langle \frac{dq_{i_1}}{dt_1} q_{i_2}(t_2)\rangle$$

However, in the quantum case, the time ordered correlation functions may be sensitive specially when two operators have the same time argument. This subtle point, is best observed when the differential operator is such that the classical counter part would correspond to the RHS of the Euler-Lagrange equations of motion which is zero.

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = 0$$

For example and for a simple harmonic oscillator, the classical theory predicts

$$\Big[\big(\frac{d}{dt_1}\big)^2 + \omega^2\Big]\langle X(t_1)X(t_2)\rangle = \Big\langle\Big(\frac{d^2X(t_1)}{dt_1^2} + \omega^2 X(t_1)\Big)X(t_2)\Big\rangle = 0$$

As we will see, the quantum theory provides a different answer.

To find the quantum version of the above equation (in general), let us consider the path integral

$$\int Dq\, e^{iS[q]} q_{i_1}(t_1)\cdots q_{i_n}(t_n).$$

Using the infinitesimal change of variables

$$q_i' = q_i + \varepsilon f_i(t)$$

we may write

$$\int Dq\, e^{iS[q]} q_{i_1}(t_1)\cdots q_{i_n}(t_n) = \int Dq'\, e^{iS[q']} q_{i_1}'(t_1)\cdots q_{i_n}'(t_n)$$
$$= \int Dq\, e^{iS[q']} q_{i_1}'(t_1)\cdots q_{i_n}'(t_n)$$

where in the last equality we have used the fact that a *shift* in the integration variables does not change the measure. Expanding up to the first order in ε, this implies

$$\int Dq\, e^{iS[q]} q_{i_1}(t_1)\cdots q_{i_n}(t_n)\Big[\int dt\, f_i(t)\big(\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}\big)\Big] = i\sum_{r=1}^{n}\int Dq\, e^{iS[q]} q_{i_1}(t_1)\cdots q_{i_{r-1}}(t_{r-1}) f_{i_r}(t_r) q_{i_{r+1}}(t_{r+1})\cdots q_{i_n}(t_n)$$

By letting $f_i(t) = \delta_{ii_0}\delta(t-t_0)$ for some i_0, t_0 we immediately get to translate these integral equations into correlation function results known as the Schwinger-Dyson equations

$$\boxed{\Big\langle \mathscr{T}\Big\{\big(\frac{\partial L}{\partial q_{i_0}} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_{i_0}}\big)_{t=t_0} q_{i_1}(t_1)\cdots q_{i_n}(t_n)\Big\}\Big\rangle = i\sum_{r=1}^{n}\delta_{i_r i_0}\delta(t_r - t_0)\Big\langle \mathscr{T}\big\{q_{i_1}(t_1)\cdots q_{i_{r-1}}(t_{r-1}) q_{i_{r+1}}(t_{r+1})\cdots q_{i_n}(t_n)\big\}\Big\rangle}$$

Of course, the operator

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}$$

is zero; what we mean by the left hand side of the Schwinger-Dyson equation is that the differentiation operators are acting on the correlation function. Back to our example of the simple harmonic oscillator, this reads

$$\Big[\big(\frac{d^2}{dt_1^2}\big) + \omega^2\Big]\Big\langle \mathscr{T}\big\{X(t_1)X(t_2)\big\}\Big\rangle = -i\delta(t_1 - t_2)$$

Symmetries of the dynamical system lead to conserved charges in the classical formalism. Since the conservation of a charge X^α is a result of the Euler-Lagrange equations of motion, we may also ask similar questions about the correlation functions involving the observable $\frac{dX^\alpha}{dt}$. Using, the Schwinger-Dyson equation, it is rather straightforward to prove the Ward identities.

$$\boxed{\Big\langle \mathscr{T}\big\{\frac{dX^\alpha}{dt}\big|_{t_0} q_{i_1}(t_1)\cdots q_{i_n}(t_n)\big\}\Big\rangle = -i\sum_{r=1}^{n}\delta(t_0 - t_r)\Big\langle \mathscr{T}\big\{\tau^\alpha_{i_r}(t_0) q_{i_1}(t_1)\cdots q_{i_n}(t_n)\big\}\Big\rangle}$$

The proof for these identities is left as an exercise.

7.5 Exercises

1. Use the Schwinger-Dyson formula and the symmetry properties of the 2-point correlation function for the simple harmoni oscillator to prove

$$\Big\langle \mathscr{T}\big\{X(t_1)X(t_2)\big\}\Big\rangle = \langle X^2(0)\rangle \cos\big(\omega(t_1 - t_2)\big) + \frac{1}{2i\omega}\sin\big(\omega|t|\big)$$

2. Define the ground state generator functional as

$$Z[\sigma] = \int Dq\, e^{iS[q]} e^{-i\int dt\,\sigma_i(t) q_i(t)}$$

show that

$$\Big\langle \mathscr{T}\big\{q_{i_1}(t_1)\cdots q_{i_n}(t_n)\big\}\Big\rangle = \frac{(i\delta)^n}{\delta\sigma_{i_1}(t_1)\cdots\delta\sigma_{i_n}(t_n)}\log(Z[\sigma])\big|_{\sigma=0}$$

3. To prove the Ward identities, using the Schwinger-Dyson equations

   a) Show that

$$\frac{dX^\alpha}{dt} = -\tau_i^\alpha \Big(\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} \Big)$$

   b) Now use the previous part and the Schwinger-Dyson equations to prove the Ward identity.

Chapter 8

Entropies and Entanglement

In this chapter we review the bare minimum essentials of the quantum information theory. The material is not needed until much further down the road when we discuss holographies and entanglements in quantum field theories.

8.1 Entropies

For a classical probability vector $P(x)$, the Renyi entropies are defined as

$$\mathtt{S}_\alpha \equiv \frac{1}{1-\alpha} \log \Big(\sum_x P^\alpha(x) \Big)$$

For a review of the properties of the Renyi entropies, check out the exercises.

The natural generalisation of these quantities to quantum mechanics is

$$\mathtt{S}_\alpha(\rho) \equiv \frac{1}{1-\alpha} \log \operatorname{Tr} \rho^\alpha$$

Our main motivation for defining these entropies is to have a way of computing the Shannon - Von Neumann entropy. This is done as

$$\begin{aligned} \mathtt{S}_1 &= -\langle \log P(x) \rangle = -\Big\langle \log \Big\{ 1 - \big[1 - P(x)\big] \Big\} \Big\rangle \\ &= \sum_{m=1}^{\infty} \frac{1}{m} \big\langle \big[1 - P(x)\big]^m \big\rangle \\ &= \sum_{m=1}^{\infty} \frac{1}{m} \sum_{n=0}^{m} \binom{m}{n} (-1)^n e^{-n\mathtt{S}_{n+1}} \end{aligned}$$

8.2 Exercises

1. By considering the limit $\alpha \to 1$, show that $\mathtt{S}_1$ is the same as the Shannon - Von Neumann entropy.

2. a) One way to measure the correlation between two systems is to compute the covariance for two observables

$$\mathrm{Cov}(O_A, O_B) \equiv \langle O_A O_B \rangle - \langle O_A \rangle \langle O_B \rangle$$

since this quantity depends on the specific choice of the observables O_A and O_B, we define the correlation measure as

$$\mathtt{C}(A:B) \equiv \max_{O_A,O_B} \frac{\mathrm{Cov}(O_A,O_B)}{||O_A||_2\,||O_B||_2}$$

Show that the optimal operators are written as $O^* = 2P^* - \mathbb{I}$ where P^* is a projector operator.

b) The total-variation (or trace) distance between two density matrices is defined as

$$\mathrm{TV}(\rho,\sigma) \equiv \frac{1}{2}\,\mathrm{Tr}(|\rho-\sigma|)$$

Prove

$$\mathtt{C}(A:B) \leq \mathrm{TV}(\rho_{AB}, \rho_A \otimes \rho_B)$$

c) ⚓ The Kullback-Leibler divergence is defined as

$$D_{KL}(\rho||\sigma) \equiv \mathrm{Tr}\left[\rho(\log\rho - \log\sigma)\right]$$

Prove **Pinsker's inequality**

$$\mathrm{TV}(\rho,\sigma) \leq \sqrt{2D_{KL}(\rho||\sigma)}$$

d) Finally, define the mutual information

$$\mathtt{I}(A:B) \equiv \mathtt{S}_1(A) + \mathtt{S}_1(B) - \mathtt{S}_1(AB)$$

and by proving

$$\mathtt{I}(A:B) = D_{KL}(\rho_{AB}||\rho_A \otimes \rho_B)$$

conclude with

$$\mathtt{C}^2(A:B) \leq 2\mathtt{I}(A:B)$$

Note that this implies the **subadditivity** property for $\mathtt{S}_1$.

$$\mathtt{S}_1(AB) \leq \mathtt{S}_1(A) + \mathtt{S}_1(B)$$

e) 🔔 By proving

$$\mathtt{I}(A:B) \leq D_{KL}(\rho_A \otimes \rho_B||\rho_{AB})$$

show that the result of the previous part is not improved by swapping ρ_{AB} and $\rho_A \otimes \rho_B$.

3. 🔔 Generalise the **Donsker-Varadhan** formula to quantum systems

$$D_{KL}(\rho||\sigma) = \max_{A=A^\dagger}\left\{\mathrm{Tr}\,\rho A - \log\mathrm{Tr}\,\sigma e^A\right\}$$

4. ⚓ Prove the two versions of **strong subadditivity** property for the Shannon - Von Neumann entropy.

   a)
$$\mathtt{I}(A:B) \leq \mathtt{I}(A:BC)$$

   b)
$$\mathtt{S}_1(A) + \mathtt{S}_1(C) \leq \mathtt{S}_1(AB) + \mathtt{S}_1(BC)$$

5. Prove the **Araki-Lieb** inequality
$$|\mathtt{S}_1(A) - \mathtt{S}_1(B)| \leq \mathtt{S}_1(AB)$$
   Hint: First purify the state ρ_{AB} by adding a third system C and then use the strong subadditivity property.

6. a) Show that the Renyi entropies are decreasing function of α by proving
$$\frac{d\mathtt{S}_\alpha(p)}{d\alpha} = -(1-\alpha)^{-2} D_{KL}(q_\alpha || p)$$
   where $q_\alpha(x) = \frac{p^\alpha(x)}{\sum_x p^\alpha(x)}$.

   b) ⚓ Use the previous result to prove the following inequalities as well
$$\frac{d}{d\alpha}(1-\alpha)\mathtt{S}_\alpha \leq 0$$
$$\frac{d}{d\alpha}(1/\alpha - 1)\mathtt{S}_\alpha \leq 0$$

   c) ⚓ Prove
$$\frac{d^2}{d\alpha^2}(1-\alpha)\mathtt{S}_\alpha \geq 0$$

   d) Show that the Renyi entropy itself is not necessarily convex. Hint: Consider a probability vector with a dominant component.

7. For a thermal state
$$\rho(\beta) = \frac{1}{Z(\beta)} e^{-\beta H}$$
   show that the Renyi entropies are given by
$$\mathtt{S}_\alpha\big(\rho(\beta)\big) = \frac{\alpha\beta}{1-\alpha}\big[F(\beta) - F(\alpha\beta)\big]$$
   where $F(\beta) \equiv -\beta^{-1} \operatorname{Tr} e^{-\beta H}$ is the free energy.

8. ⚓ Consider a random, normalized pure state $|\psi\rangle$ in the N - particle Hilbert space $\mathscr{H} = \mathscr{H}_0^N$ and consider the entanglement entropy between the first xN particles and the rest of the system as a random variable

$$A \equiv \mathtt{S}_1\Big(\mathrm{Tr}_{(1-x)N} \, |\psi\rangle \langle\psi| \Big)$$

Show that in the large N limit, the following concentration of measure occurs.

$$\frac{A}{N \log \dim \mathscr{H}_0} \to \min(x, 1-x)$$

Part III

Quantum Field Theories on Flat Space-Times

Chapter 9

Quantum Klein-Gordon Fields

Our first quantum field theory, will be that corresponding to the free scalar field theory with the Lagrangian

$$\mathcal{L} = -\frac{1}{2}\sum_a \left[\eta^{\mu\nu}(\partial_\mu\phi_a)(\partial_\nu\phi_a) + m^2\phi_a^2\right]$$

This is a second order function of the fields and their time derivatives, therefore we expect to be able to solve the model exactly by finding the normal modes.

The translation invariance of the model, persuades us to use the Fourier transform

$$\tilde{\phi}_a(\mathbf{k}, t) = (2\pi)^{-n/2}\int d\mathbf{x}\,\phi_a(\mathbf{x}, t)e^{-i\mathbf{k}.\mathbf{x}}$$

$$\tilde{\pi}_a(\mathbf{k}, t) = (2\pi)^{-n/2}\int d\mathbf{x}\,\pi_a(\mathbf{x}, t)e^{-i\mathbf{k}.\mathbf{x}}$$

At first, this looks promising and block diagonalizes the Hamiltonian as

$$H = \frac{1}{2}\sum_a \int d\mathbf{x}\left[\pi^2 + ||\boldsymbol{\nabla}\phi||_2^2 + m^2\phi^2\right]$$

$$= \frac{1}{2}\sum_a \int d\mathbf{k}\left[\tilde{\pi}_a\tilde{\pi}_a^\dagger + \omega^2(\mathbf{k})\tilde{\phi}_a\tilde{\phi}_a^\dagger\right]$$

with

$$\omega(\mathbf{k}) \equiv +\sqrt{||\mathbf{k}||_2^2 + m^2}$$

However, some peculiarities arise due to the introduction of complex coefficients $e^{\pm i\mathbf{k}.\mathbf{x}}$. First of all the $\tilde{\phi}_a$ and $\tilde{\pi}_a$ are not Hermitian operators; in fact

$$\tilde{\phi}_a^\dagger(\mathbf{k}, t) = \tilde{\phi}_a(-\mathbf{k}, t); \quad \tilde{\pi}_a^\dagger(\mathbf{k}, t) = \tilde{\pi}_a(-\mathbf{k}, t)$$

Second, $\tilde{\phi}_a$, $\tilde{\pi}_a$ are not conjugate operators, rather

$$[\tilde{\phi}_a(\mathbf{k}, t), \tilde{\pi}_b(\mathbf{k}', t)] = (2\pi)^{-n}\int d\mathbf{x}\,d\mathbf{x}'\,e^{-i(\mathbf{k}.\mathbf{x}+\mathbf{k}'.\mathbf{x}')}[\phi_a(\mathbf{x}, t), \pi_b(\mathbf{x}', t)] = i\delta(\mathbf{k}+\mathbf{k}')\delta_{ab}$$

$$[\tilde{\phi}_a(\mathbf{k}, t), \tilde{\phi}_b(\mathbf{k}', t)] = (2\pi)^{-n}\int d\mathbf{x}d\mathbf{x}'\,e^{-i(\mathbf{k}.\mathbf{x}+\mathbf{k}'.\mathbf{x}')}[\phi_a(\mathbf{x}, t), \phi_b(\mathbf{x}', t)] = 0$$

$$[\tilde{\pi}_a(\mathbf{k},t),\tilde{\pi}_b(\mathbf{k}',t)] = (2\pi)^{-n}\int d\mathbf{x}d\mathbf{x}'\, e^{-i(\mathbf{k}.\mathbf{x}+\mathbf{k}'.\mathbf{x}')}[\pi_a(\mathbf{x},t),\pi_b(\mathbf{x}',t)] = 0$$

In other words, $\tilde{\phi}_a$ are conjugate to $\tilde{\pi}_a^\dagger$.

Despite all this, miracle happens when we define the creation and annihilation operators as

$$a_a(\mathbf{k},t) \equiv \omega(\mathbf{k})\tilde{\phi}_a(\mathbf{k},t) + i\tilde{\pi}_a(\mathbf{k},t)$$

Do they satisfy the commutation relations that we desire? Indeed!

$$[a_a(\mathbf{k},t),a_b(\mathbf{k}',t)] = [a_a^\dagger(\mathbf{k},t),a_b^\dagger(\mathbf{k}',t)] = 0; \quad [a_a(\mathbf{k},t),a_b^\dagger(\mathbf{k}',t)] = 2\omega(\mathbf{k})\delta_{ab}\delta(\mathbf{k}-\mathbf{k}')$$

The only problem here seems to be a bad choice of normalization in defining the creation and annihilation operators. Normally, we would want to have $\delta_{ab}\delta_{\mathbf{k}\mathbf{k}'}$ as the RHS. To compensate for this abnormal normalisation choice, we use the following rule of thumb: Any product of $2n$ creation/annihilation operators, must be accompanied by n factors of $\frac{d\mathbf{k}}{2\omega(\mathbf{k})}$, whatever factor that remains is the actual, legitimate weight of the product. The reader now understands the reason behind our specific normalisation choice since this is proportional to the covariant measure over the manifold $k_\mu k^\mu = -m^2$, $k^0 > 0$. In fact

$$\int_{k^0>0}\delta(k_\mu k^\mu + m^2)\,dk = \int\frac{d\mathbf{k}}{2\omega(\mathbf{k})}$$

Now let us write down the Hamiltonian. Using

$$\tilde{\phi}_a(\mathbf{k},t) = \frac{a_a(\mathbf{k},t)+a_a^\dagger(-\mathbf{k},t)}{2\omega(\mathbf{k})}; \quad \tilde{\pi}_a(\mathbf{k},t) = \frac{a_a(\mathbf{k},t)-a_a^\dagger(-\mathbf{k},t)}{2i}$$

the Hamiltonian may be re-written as

$$H = \sum_a\int\frac{d\mathbf{k}}{2\omega(\mathbf{k})}\,\omega(\mathbf{k})a_a^\dagger(\mathbf{k},t)a_a(\mathbf{k},t) + \frac{1}{2}\sum_a\int d\mathbf{k}\,\omega(\mathbf{k})\delta(\mathbf{k}-\mathbf{k})$$

The first term is in the shape we love: this is the $\sum_{\mathrm{i}}\omega_i a_i^\dagger a_i$ part of a harmonic oscillator Hamiltonian written in a covariant manner. This shows us that each mode $\mathbf{k}$ has frequency (surprise:) $\omega(\mathbf{k})$. The second term is among the most evil things one could write down; the integrand is infinite, the bounds are infinite, etc. Although we will eventually drop this term as an irrelevant, constant term in the Hamiltonian that only elevates the ground state, let us take a closer look first. Before anything, note that this may be re-written as

$$\frac{1}{2}\sum_a\sum_{\mathbf{k}}\omega(\mathbf{k}) = \frac{1}{2}\sum_{\text{modes}}\omega_{\text{mode}}$$

which is still as infinite as it was, but now looks like something that we must have foreseen all along. On the other hand, if we limit our space to a box of volume V, the horrible δ term becomes an innocent V. This means, whatever remains is the vacuum energy *density*

$$\rho = \frac{1}{2}\sum_a\int d\mathbf{k}\,\omega(\mathbf{k})$$

This still infinite energy density, may be made finite by some next generation theory that puts a limit on how small the wavelength and therefore how large the $\mathbf{k}$ vector could get. (Say by limiting the number of spatial *points*, etc.)

In order to know the Hilbert space better, we start by diagonalising the positive semi definite number operator

$$N \equiv \sum_a N_a; \quad N_a \equiv \int\frac{d\mathbf{k}}{2\omega(\mathbf{k})}\mathcal{N}_a(\mathbf{k},t); \quad \mathcal{N}_a(\mathbf{k},t) \equiv a_a^\dagger(\mathbf{k},t)a_a(\mathbf{k},t)$$

Because of the ladder structure $[N, a_a(\mathbf{k})] = -a_a(\mathbf{k})$, we know that the whole Hilbert space must end in some *vacuum* subspace $\mathscr{V}$ such that

$$\forall \ |\Omega\rangle \in \mathscr{V}, \ \ a \in [K], \ \ \mathbf{k} \quad a_a(\mathbf{k})\,|\Omega\rangle = 0$$

If we define the Fock space $\mathscr{F}$ to be the Hilbert space with the basis

$$|(b_1, \mathbf{k}_1), \cdots, (b_n, \mathbf{k}_n)\rangle \equiv \Big(\prod_{i=1}^{n} \frac{a^\dagger_{b_i}(\mathbf{k}, t)}{\sqrt{2\omega(\mathbf{k}_i)}}\Big)\,|\Omega\rangle$$

for some $|\Omega\rangle$, then we find that the Hilbert space is (nothing more than)

$$\mathscr{H} = \mathscr{F} \otimes \mathscr{V}$$

The energy-momentum four vector is the operator

$$P^\mu \equiv \int d\mathbf{x}\, T^{0\mu} = (H, \mathbf{P})$$

with

$$\mathbf{P} \equiv -\sum_a \int \pi_a \boldsymbol{\nabla}\phi_a \, d\mathbf{x} = \sum_a \int \frac{d\mathbf{k}}{2\omega(\mathbf{k})}\, \mathbf{k}\mathcal{N}_a(\mathbf{k}, t)$$

So far, we have restricted our discussion of operators to a single space-like surface with constant time. The generalisation to different space sheets is straightforward. We begin with the creation/annihilation operators

$$\frac{d}{dt} a_a(\mathbf{k}, t) = i[H, a_a(\mathbf{k}, t)] = -i\omega(\mathbf{k}) a_a(\mathbf{k}, t)$$

This is a great result: the most interesting operators have an equal number of as and $a^\dagger$s and therefore stay the same through time which is equivalent to commuting with the Hamiltonian.

The fields evolve as

$$\phi_a(x^\mu) = (2\pi)^{-n/2} \int_{k^0>0} dk\, \delta(k_\alpha k^\alpha + m^2) a_a(k) e^{i\eta_{\alpha\beta} x^\alpha k^\beta} \quad + \quad h.c.$$

Note that this manifestly satisfies the Klein-Gordon equation

$$(\partial_\mu \partial^\mu - m^2)\phi_a = 0$$

Now Imagine Alice performing some local operation on the fields confined to a space-time region A. This may be described using a unitary operator $U = e^{iO_A}$ where O_A is a local observable.

$$O_A = O_A\Big(\phi_a(A), \pi_a(A)\Big)$$

Meanwhile, Bob is measuring some other local observable at space-time region B which is causally disconnected from A.

$$O_B = O_B\Big(\phi_a(B), \pi_a(B)\Big)$$

For our theory to be causal, we need

$$\langle\psi|U^\dagger O_B U|\psi\rangle = \langle\psi|O_B|\psi\rangle$$

which is equivalent to

$$[\phi_a(x), \phi_b(y)] \overset{!}{=} 0; \;\; [\phi_a(x), \pi_b(y)] \overset{!}{=} 0 \quad \forall\ (x-y)^2 > 0$$

In fact, we only need the first condition since it would immediately imply

$$[\phi_a(x), \pi_b(y)] = \Big[\phi_a(x), \frac{\partial}{\partial y^0}\phi_b(y)\Big] = \frac{\partial}{\partial y^0}[\phi_a(x), \phi_b(y)] = 0; \quad \forall\ (x-y)^2 > 0$$

Now for the field-field commutator we first set $y = 0$ then

$$[\phi_a(x), \phi_b(0)] = 2i\delta_{ab}(2\pi)^{-n} \int_{k^0>0} dk\, \delta(k^2 + m^2) \sin(k_\mu x^\mu)$$

which clearly vanishes for $x^2 > 0$ and therefore our theory is a causal one.

9.1 The Correlation Functions and the Wick's Theorem

In order to find all the observable expected values for the Klein-Gordon field theory, it suffices to have access to the vacuum correlation functions.

$$\langle\Omega| \prod_i \phi_{a_i}(x_i) |\Omega\rangle$$

Clearly, we only need to consider the correlation functions for a single type of fields, hence we drop the a_i indices. It also costs no generality to consider only the time ordered correlation functions

$$\langle\Omega| \mathscr{T}\big\{\phi(x_1)\cdots\phi(x_n)\big\} |\Omega\rangle$$

since we have already computed the field commutators and are therefore able to move from any arbitrary order towards the time order.

Our strategy for evaluating the correlation functions is as follows:

- Write down the field operators in terms of creation/annihilation operators.
- Move the annihilation (creation) operators to the right (left) and get extra terms from the canonical commutation relations.
- Simplify using $a|\Omega\rangle = 0$ to get the final result.

This strategy motivates us to define the *normal* ordering for a product of operators. In a chain of creation/annihilation operators, the normal ordering moves all the annihilation operators to the right and the creation operators to the left. For example

$$\mathscr{N}\big\{a_1 a_2^\dagger a_3 a_4 a_5^\dagger\big\} = a_2^\dagger a_5^\dagger a_1 a_3 a_4$$

To carry out the first step, we define

$$\phi^+(x) \equiv (2\pi)^{-n/2} \int_{k^0>0} dk\, \delta(k^2+m^2) e^{i\eta_{\alpha\beta}k^\alpha x^\beta} a(k); \quad \phi^-(x) = \phi^{+\dagger}(x)$$

Therefore a time ordered product of field operators becomes

$$\mathscr{T}\big\{\phi(x_1)\cdots\phi(x_n)\big\} = (\phi^+(x_1) + \phi^-(x_1))\cdots(\phi^+(x_n) + \phi^-(x_n))$$

Note that without loss of generality, we have assumed that the labellings are consistent with the time ordering.

The next step would be to move creation operators to the left. While doing this, we would get terms in the form

$$D_F(x-y) \equiv \begin{cases} [\phi^+(x), \phi^-(y)] & x^0 \geq y^0 \\ [\phi^+(y), \phi^-(x)] & y^0 \geq x^0 \end{cases}$$

The space-time function $D_F(x)$ is called the Feynman propagator. We will deal with it at the end of this section.

As an example we may write

$$\mathscr{T}\big\{\phi(x)\phi(y)\big\} = \mathscr{N}\big\{\phi(x)\phi(y)\big\} + D_F(x-y)$$

$$\mathscr{T}\big\{\phi(x)\phi(y)\phi(z)\big\} = \mathscr{N}\big\{\phi(x)\phi(y)\phi(z)\big\} + \mathscr{N}\big\{\phi(x)\big\}D_F(y-z) + \mathscr{N}\big\{\phi(y)\big\}D_F(x-z) + \mathscr{N}\big\{\phi(z)\big\}D_F(x-y)$$

In general, one can see that Wick's theorem holds[1]

$$\mathscr{T}\big\{\phi(x_1)\cdots\phi(x_n)\big\} = \sum_{\substack{\text{all possible}\\ \text{contractions}}} \Big(\prod_{i\sqcap j} D_F(x_i - x_j)\Big)\mathscr{N}\big\{\text{all uncontracted operators}\big\}$$

Here, by contraction, I mean choosing a pair of field operators and removing them from the operator product chain. You can contract no pairs of operators or contract all the operators (in case there is an even number of them) to get a null product chain. Whatever pair of operators that you contract (a contracted pair is here denoted by $i \sqcap j$) results in a Feynman propagator factor.

Finally, in the last step, when evaluating the expected value in the vacuum state, the normal ordering rids us of all terms with remaining, uncontracted field operators and we get

$$\langle\Omega|\, \mathscr{T}\big\{\phi(x_1)\cdots\phi(x_n)\big\}\,|\Omega\rangle = \sum_{\substack{\text{full}\\ \text{contractions}}} \prod_{i\sqcap j} D_F(x_i - x_j)$$

As a last example

$$\langle\Omega|\, \mathscr{T}\big\{\phi(x_1)\cdots\phi(x_4)\big\}\,|\Omega\rangle = D_F(x_1-x_2)D_F(x_3-x_4) + D_F(x_1-x_3)D_F(x_2-x_4) + D_F(x_1-x_4)D_F(x_2-x_3)$$

Now let us focus on the Feynman propagator; by definition this is

$$D_F(t,\mathbf{x}) = (2\pi)^{-n}\int \frac{d\mathbf{k}}{2\omega(\mathbf{k})}\, e^{i\,\mathrm{sgn}(t)\big[\mathbf{k}.\mathbf{x} - \omega(\mathbf{k})t\big]}$$

$$= (2\pi)^{-n}\int \frac{d\mathbf{k}}{2\omega(\mathbf{k})}\, \cos(\mathbf{k}.\mathbf{x})e^{-i\omega(\mathbf{k})|t|}$$

In appendix A, we have further discussed the space-time values of the Feynman propagator but for now we are more interested in its Fourier transform.

[1] Here, I assume $\mathscr{N}\{\} = 1$. You could say this is an exception in the definition of time and normal ordering operations, since one could write

$$\mathscr{N}\{1\} = \mathscr{N}\{[a, a^\dagger]\} = 0$$

In fact you shouldn't take $\mathscr{N}$ and $\mathscr{T}$ too seriously; they are only here to make our equations look better ie. shorter.

Using theorems of complex integration it is straightforward to prove the following useful identity

$$-i\pi e^{-i|\lambda|} = \lim_{\varepsilon\to 0^+}\int_{-\infty}^{+\infty} dz\,\frac{e^{-i\lambda z}}{z^2-1+i\varepsilon}$$

which helps us write

$$\begin{aligned} D_F(t,\mathbf{x}) &= (2\pi)^{-(n+1)} i\int \frac{d\mathbf{k}}{\omega(\mathbf{k})}\,e^{i\mathbf{k}.\mathbf{x}}\lim_{\varepsilon\to 0}\int_{-\infty}^{+\infty} dz\,\frac{e^{-i\omega(\mathbf{k})tz}}{z^2-1+i\varepsilon} \\ &= (2\pi)^{-(n+1)}\lim_{\varepsilon\to 0}\int dk\,e^{ik_\mu x^\mu}\,\frac{i}{k^2+m^2-i\varepsilon} \end{aligned}$$

We encode on this in the symbolic equation

$$\tilde{D}_F(k_\mu) = \frac{i(2\pi)^{-(n+1)/2}}{k^2+m^2-i\varepsilon}$$

Exercises

1. The Klein-Gordon Lagrangian for m independent fields $a = 1, 2, \cdots, m$ is invariant under the $SO(m)$ transformations

$$\phi_a \to \phi_a + \varepsilon A_{ab}\phi_b; \quad A_{\{ab\}} = 0$$

These are called the internal symmetries of the field theory. Find the corresponding $\binom{m}{2}$ conserved currents. $\big($*Ans.* $j^\mu(A) = A_{ab}\phi_a\partial^\mu\phi_b$ $\big)$

2. Repeat what we did for the Lagrangian

$$\mathcal{L} = -\frac{1}{2}\eta^{\alpha\beta}\left[m^2 A_\alpha A_\beta + \eta^{\mu\nu}(\partial_\mu A_\alpha)(\partial_\nu A_\beta)\right]$$

and find *all* the conserved currents. Compare your results with the previous exercise.

3. ⚓ Repeat the results of this chapter for the Lagrangian

$$\mathcal{L} = -\frac{1}{2}K^{ab}\eta^{\mu\nu}\partial_\mu\phi_a\partial_\nu\phi_b - \frac{1}{2}m^2\phi_a\phi_a$$

where K^{ab} is a constant, positive definite matrix.

4. Let $\mathbf{X}$ be a vector of N centerred random variables (that is $\langle X_i\rangle = 0$). Assume that the covariance matrix is given as

$$C_{ij} = \langle X_i X_j\rangle$$

If the higher order moments satisfy

$$\langle X_{i_1}\cdots X_{i_n}\rangle = \sum_{\substack{\text{full}\\ \text{contractions}\\ \text{of } i_1\cdots i_n}} \prod_{i\sqcap j} C_{ij}$$

then show that the vector $\mathbf{X}$ is normally distributed.

Hint: First, show that without loss of generality, it is possible to take $C_{ij} = \delta_{ij}$ and then show that for the corresponding, *independent* normal distribution, the conditions are satisfied.

5. In light of the mode function formalism, write the scalar field as

$$\phi(\mathbf{x},t) = (2\pi)^{-n/2}\int \frac{d\mathbf{k}}{\sqrt{2\omega(\mathbf{k})}} a(\mathbf{k}) e^{i\mathbf{k}.\mathbf{x}} v(\mathbf{k},t) + h.c.$$

Explain the integral measure, assuming that a has the same normalisation as in the chapter. Let $A(t)$ be the quantity

$$A(t) \equiv \int d\mathbf{x} W(\mathbf{x})\phi(\mathbf{x},t)$$

Show that the ground state fluctuation of this quantity is given by

$$\langle 0|\, A^2(t)\, |0\rangle = \int d\mathbf{k} |v(\mathbf{k},t)|^2 \, |\tilde{W}(\mathbf{k})|^2$$

Assuming isotropy in both W and v, write it down as

$$\langle 0|\, A^2(t)\, |0\rangle = \int_0^\infty d\log(k)\; \mathcal{P}(k,t)\, |\tilde{W}(k)|^2$$

with

$$\mathcal{P}(k,t) = \frac{n\pi^{n/2}}{\Gamma(n/2+1)} k^n |v(k,t)|^2$$

Finally, show that for the time-independent Minkowski space-time discussed in the chapter

$$\mathcal{P} = 2\pi\frac{k^3}{\omega}$$

6. a) Define

$$\bar{\phi}_\mu(x) = \mu^{n+1}\int dy\, w(\mu y)\, \phi(x+\mu y)$$

Show that

$$\langle\Omega|\, \bar{\phi}_\mu(x)\bar{\phi}_\mu(x)\, |\Omega\rangle = 2\pi\int \frac{d\mathbf{k}}{2\omega(\mathbf{k})}\; |\tilde{w}\,(k/\mu)\,|^2$$

b) For

$$w(x) = (2\pi)^{-n/2}\,\delta(x^0)\,\exp\left(-\frac{\mathbf{x}^2}{2}\right)$$

show that

$$\langle\Omega|\, \bar{\phi}_\mu(x)\bar{\phi}_\mu(x)\, |\Omega\rangle = (2\pi)^{1-n}\int_0^\infty \frac{k^2 e^{-k^2/\mu^2}\, dk}{\sqrt{m^2+k^2}}$$

and find the asymptotic approximations for small and large μ values.

$$\mu \ll 1 \;:\quad \frac{\sqrt{\pi}(2\pi)^{1-n}}{4m}\mu^3$$

$$\mu \gg 1 \;:\quad \frac{1}{2}(2\pi)^{1-n}\mu^2$$

Chapter 10

The $\lambda\phi^4$ Theory

We now consider our first interacting quantum field theory called the $\lambda\phi^4$ - theory. The Lagrangian density is

$$\mathcal{L} = -\frac{1}{2}\eta^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi) - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4$$

Assuming $\lambda \ll 1$, it sounds reasonable to use the perturbation theory that we developed in previous chapters. In this regard, the interaction Hamiltonian is

$$H_1 = \frac{\lambda}{4!}\int d\mathbf{x}\,\phi^4(\mathbf{x})$$

Although we are going to use the interaction picture operators for the rest of this chapter (and perhaps more), we drop the superscript: $O^0 \to O$.

Like the *free* Klein-Gordon theory, we are interested in evaluating the N - point correlation functions.

$$\langle\Omega|\,\mathscr{T}\big\{\phi(x_1)\cdots\phi(x_n)\big\}\,|\Omega\rangle$$

But we already know that this is written as

$$\langle\Omega|\,\mathscr{T}\big\{\phi(x_1)\cdots\phi(x_n)\big\}\,|\Omega\rangle = \frac{\langle\Omega_0|\,\mathscr{T}\big\{\phi(x_1)\cdots\phi(x_n)S\big\}\,|\Omega_0\rangle}{\langle\Omega_0|\,S\,|\Omega_0\rangle}$$

10.1 The Diagrammatic Notation

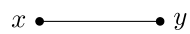
Let us consider the 2-point correlation function. Using Wick's theorem, we may expand this in λ as

$$\langle\Omega|\,\mathscr{T}\big\{\phi(x)\phi(y)\big\}\,|\Omega\rangle = \frac{D_F(x-y) - \frac{i\lambda}{4!}\int dz\,\big[3D_F(x-y)D_F^2(z-z) + 12D_F(x-z)D_F(y-z)D_F(z-z)\big] + \cdots}{1 - 3\frac{i\lambda}{4!}\int dz\,D_F^2(z-z) + \cdots}$$

Our current notation is now clearly getting repetitive, it is definitely easier to describe each term in the expansion with words than to write it down explicitly and since a picture is worth a thousand words we are tempted to use a diagrammatic notation. First of all, to denote a field operator on space-time position x, we simply use a node.

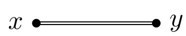
$$= \phi(x)$$

The Feynman propagators connect space-time field operators and are therefore denoted by lines such as

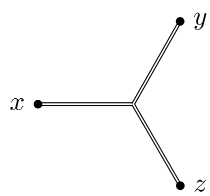


$$= D_F(x-y) = \langle\Omega_0|\,\mathscr{T}\big\{\phi(x)\phi(y)\big\}\,|\Omega_0\rangle$$

The perturbed correlation functions are denoted by a double line connection, for example

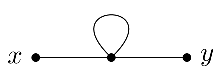


$$= \langle\Omega|\,\mathscr{T}\big\{\phi(x)\phi(y)\big\}\,|\Omega\rangle = \frac{\langle\Omega_0|\,\mathscr{T}\big\{\phi(x)\phi(y)S\big\}\,|\Omega_0\rangle}{\langle\Omega_0|\,S\,|\Omega_0\rangle}$$

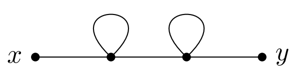


$$= \langle\Omega|\,\mathscr{T}\big\{\phi(x)\phi(y)\phi(z)\big\}\,|\Omega\rangle = \frac{\langle\Omega_0|\,\mathscr{T}\big\{\phi(x)\phi(y)\phi(z)S\big\}\,|\Omega_0\rangle}{\langle\Omega_0|\,S\,|\Omega_0\rangle}$$

Finally, we know that every power of $-i\frac{\lambda}{4!}$ adds 4 field operators at some dummy space-time position z which is to be integrated over. There are also combinatorial factors such as 3 or 12 that correspond to the number of ways that such a contraction of field operators appears. Since the $-i\frac{\lambda}{4!}$, the space-time integral and the combinatorial factor are all determined by the shape of the diagram, we simply do not write them. Therefore and for example we have



$$= \frac{1}{1!}(12)\frac{-i\lambda}{4!}\int dz\, D_F(x-z)D_F(y-z)D_F(z-z)$$



$$= \frac{1}{2!}(288)\Big(\frac{-i\lambda}{4!}\Big)^2\int dz_1 \mathrm{d}z_2\, D_F(x-z_1)D_F(z_1-z_1)D_F(z_1-z_2)D_F(z_2-z_2)D_F(z_2-y)$$

We will denote the combinatorial factor for a diagram D with $\#(D)$. Now consider the diagram D_1D_2 consisting of two disconnected parts D_1 and D_2 with n_1 and n_2 interaction vertices respectively. We have

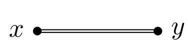
$$\frac{\#(D_1D_2)}{(n_1+n_2)!} = \binom{n_1+n_2}{n_1}\frac{\#(D_1)\#(D_2)}{(n_1+n_2)!} = \frac{\#(D_1)\#(D_2)}{n_1!n_2!}$$

This means that the final, numerical value of a diagram is the product of the numerical value of its connected components.

$$\mathrm{val}(D_1D_2) = \mathrm{val}(D_1)\,\mathrm{val}(D_2)$$

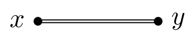
10.2 The 2-Point Correlation Function

Armed with our new, efficient notation it is now straight forward to write down an expression for the perturbed 2-point correlation function

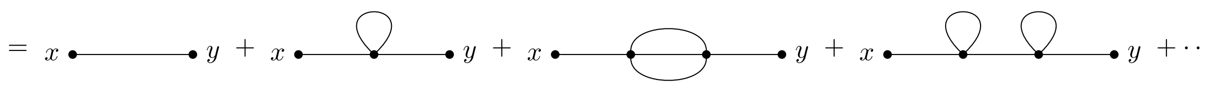


$$= \frac{\sum \text{all possible diagrams}}{1+\cdots}$$

It is possible to factor out the exact same term as in the denominator from the numerator, leaving only the connected diagrams

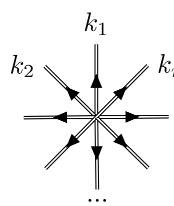


$= \sum$ all connected diagrams

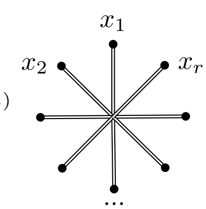


10.3 Fourier Space Diagrams

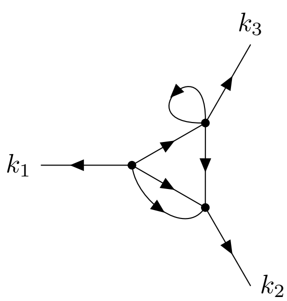
Just like the unperturbed case, we would sometimes prefer working with the Fourier transforms



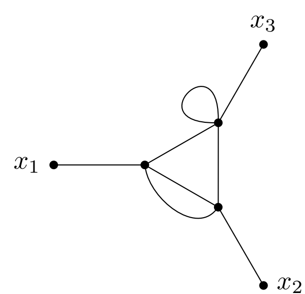
$$\equiv (2\pi)^{-r(n+1)/2} \int dx_1 \cdots dx_r \, e^{-i(k_1.x_1 + \cdots k_r.x_r)}$$



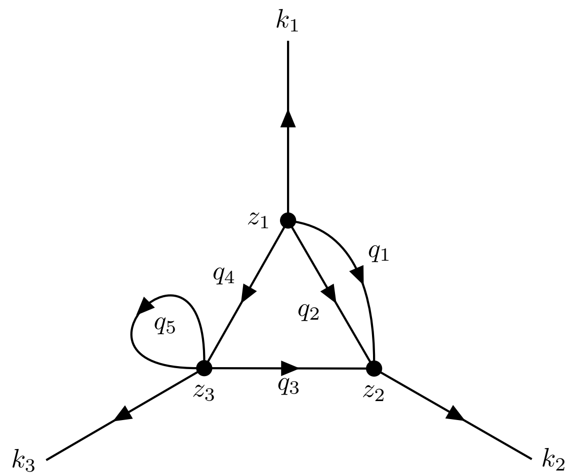
Due to the linearity of Fourier transforms, each diagram is mapped to a corresponding diagram in the Fourier domain. The $\frac{-i\lambda}{4!}$ factors, the exponential factorials and the combinatorial factor $\#(D)$ also remain the same. For example



$$\equiv (2\pi)^{-3(n+1)/2} \int dx_1 dx_2 dx_3 \, e^{-i(k_1.x_1 + k_2.x_2 + k_3.x_3)}$$



It remains to evaluate the *nuclear* value of each diagram. We do this for the example above and generalise the rules based on our observations. The strategy is to replace the propagators with their Fourier transforms. If we label the dummy indices as below



The integral we are dealing with, stripped of the extra factors is written as

$$(2\pi)^{-3(n+1)/2}\int dx_1dx_2dx_3dz_1dz_2dz_3\, e^{-i(k_1.x_1+k_2.x_2+k_3.x_3)}(2\pi)^{-8(n+1)/2}\int dq_1dq_2dq_3dq_4dq_5dk_1'dk_2'dk_3'$$
$$e^{iq_1.(z_2-z_1)}e^{iq_2.(z_2-z_1)}e^{iq_3.(z_2-z_3)}e^{iq_4.(z_3-z_1)}e^{iq_5.(z_3-z_3)}e^{ik_1'.(x_1-z_1)}e^{ik_2'.(x_2-z_2)}e^{ik_3'.(x_3-z_3)}$$
$$\tilde{D}_F(k_1')\tilde{D}_F(k_2')\tilde{D}_F(k_3')\tilde{D}_F(q_1)\tilde{D}_F(q_2)\tilde{D}_F(q_3)\tilde{D}_F(q_4)\tilde{D}_F(q_5)$$
$$=(2\pi)^{3(n+1)/2}(2\pi)^{-8(n+1)/2}\tilde{D}_F(k_1)\tilde{D}_F(k_2)\tilde{D}_F(k_3)\int dz_1dz_2dz_3dq_1dq_2dq_3dq_4dq_5$$
$$e^{iq_1.(z_2-z_1)}e^{iq_2.(z_2-z_1)}e^{iq_3.(z_2-z_3)}e^{iq_4.(z_3-z_1)}e^{iq_5.(z_3-z_3)}e^{-ik_1.z_1}e^{-ik_2.z_2}e^{-ik_3.z_3}\tilde{D}_F(q_1)\tilde{D}_F(q_2)\tilde{D}_F(q_3)\tilde{D}_F(q_4)\tilde{D}_F(q_5)$$
$$=(2\pi)^{3(n+1)/2}(2\pi)^{-8(n+1)/2}(2\pi)^{3(n+1)}\delta(k_1+k_2+k_3)\tilde{D}_F(k_1)\tilde{D}_F(k_2)\tilde{D}_F(k_3)\int dq_1dq_2dq_5$$
$$\tilde{D}_F(q_1)\tilde{D}_F(q_1)\tilde{D}_F(k_2-q_1-q_2)\tilde{D}_F(-k_1-q_1-q_2)\tilde{D}_F(q_5)$$

The same procedure applies to any diagram with translation invariant and even propagators. Since the final term is always proportional to the $\delta(k_1+\cdots+k_r)$, we will label the external k vectors in a manner that they add up to zero and then focus only on the coefficient of the delta function. In general, to evaluate this coefficient, we need to label internal wave vectors in a way that respects conservation of energy-momentum at each vertex, replace each propagator with its Fourier transform and then integrate over the undetermined momenta. The overall $(2\pi)^{(n+1)/2}$ power is given by $2\times\mathcal{V}_{int.}-\mathcal{E}_{int.}$; where $\mathcal{V}_{int.}$ is the number of internal interaction vertices and $\mathcal{E}_{int.}$ is the number of internal propagator edges, sometimes called *virtual particles.*

10.4 The Vacuum Bubbles

10.5 Exercises

1. ⚓ Consider a single simple harmonic oscillator.

$$H=\frac{p^2}{2}+\frac{x^2}{2}$$

a) Calculate the two point correlation function for the ground state

$$D(t) \equiv \langle g_0| \mathscr{T}\left\{x(0)x(t)\right\} |g_0\rangle$$

b) For the perturbed Hamiltonian

$$H = \frac{p^2}{2} + \frac{x^2}{2} + \frac{\beta}{3!}x^3 + \frac{\lambda}{4!}x^4$$

use the diagrammatic method to calculate the width of the perturbed ground state, $\langle x^2\rangle_g$ up to second order in β and λ

c) Consider a finite number of independent harmonic oscillators and then consider the perturbation

$$H = \sum_a \left(\frac{p_a^2}{2} + \frac{x_a^2}{2}\right) - \sum_a \mu_a x_a + \frac{1}{2}\sum_{ab} \alpha_{ab} x_a x_b$$

use the diagrammatic method to find the exact covariance matrix for the perturbed ground state: $\langle x_a x_b\rangle_g$

Chapter 11

From Theory to Experiment

In this chapter we try to translate our previous, abstract calculations to tangible, experimental predictions.

11.1 The LSZ Reduction Formula

Our first step towards computing experimental quantities is to find transition amplitudes between freely moving particle states

$$|i\rangle \equiv |\mathbf{k}_1, \mathbf{k}_2, \cdots, \mathbf{k}_r\rangle$$

and

$$|f\rangle \equiv \left|\mathbf{k}'_1, \mathbf{k}'_2, \cdots, \mathbf{k}'_s\right\rangle$$

after the initial state is exposed to the interacting Hamiltonian dynamics for a long time. Once again, we assume that the interaction Hamiltonian is adiabatically turned on and off through the adiabatic function, $\sigma(t)$, that we introduced before. The amplitude is then given by

$$\frac{\exp\left(i(\text{future} - \text{past})\sum_{j=1}^{s}\omega(\mathbf{k}'_j)\right)}{\langle\Omega_0|\, V(\text{future}, \text{past})\,|\Omega_0\rangle}\langle\Omega_0|\Big(\prod_{j=1}^{s}\frac{a(\text{future}, \mathbf{k}'_j)}{\sqrt{2\omega(\mathbf{k}'_j)}}\Big)V(\text{future}, \text{past})\Big(\prod_{j=1}^{r}\frac{a^\dagger(\text{past}, \mathbf{k}_j)}{\sqrt{2\omega(\mathbf{k}_j)}}\Big)|\Omega_0\rangle$$

where

$$V(b, a) \equiv \mathscr{T}\Big\{\frac{-i\lambda}{4!}\int_a^b dt\int d\mathbf{x}\,\phi^4(t, \mathbf{x})\Big\}.$$

The phase factor on the left is present to rid us from the irrelevant dynamical phases. Now let us define the modified annihilation operators

$$\alpha(t, \mathbf{k}) \equiv e^{i\omega(\mathbf{k})(t-\text{past})}V^\dagger(t, \text{past})a(\text{past}, \mathbf{k})V(t, \text{past}).$$

Using this, the transition amplitude is written as

$$\mathcal{A}\Big[\,|i\rangle \to |f\rangle\,\Big] = \frac{\langle\Omega_0|\Big(\prod_{j=1}^{s}\frac{\alpha(\text{future},\mathrm{k}'_j)}{\sqrt{2\omega(\mathbf{k}'_j)}}\Big)\Big(\prod_{j=1}^{r}\frac{\alpha^\dagger(\text{past},\mathbf{k}_j)}{\sqrt{2\omega(\mathrm{k}_j)}}\Big)|\Omega_0\rangle}{\langle\Omega_0|\, S\,|\Omega_0\rangle}$$

At t = past, the αs are the same as the good old annihilation operators, to find their value in the future, we integrate their time derivative

$$\alpha(\text{future}, \mathbf{k}) = \alpha(\text{past}, \mathbf{k}) + \int_{\text{past}}^{\text{future}} dt\,\dot{\alpha}(t, \mathbf{k})$$

$$= \alpha(\text{past}, \mathbf{k}) + i\int dt\, e^{i\omega(\mathbf{k})(t-\text{past})}V^\dagger(t, \text{past})[H_{int.}, a(t, \mathbf{k})]V(t, \text{past})$$

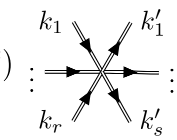
$$= \alpha(\text{past}, \mathbf{k}) - i(2\pi)^{-n/2}\int dx\, e^{-ik_\mu x^\mu}\Big(\frac{\lambda\phi_H^3}{3!}\Big)$$
$$= \alpha(0, \mathbf{k}) - i(2\pi)^{-n/2}\int dx\, e^{-ik_\mu x^\mu}(\Box - m^2)\phi_H(x)$$
$$= \alpha(0, \mathbf{k}) + i\sqrt{2\pi}(k^2 + m^2)\tilde{\phi}_H(k)$$

where $\phi_H(x) = V^\dagger(t)\phi(x)V(t)$ and the penultimate equality is a direct implication of the equations of motion. Neglecting the case of *unscattered* momenta (ie. $\mathbf{k}_j = \mathbf{k}'_j$), we can unambiguously re-write the transition amplitude as

$$\mathcal{A}\Big[\,|i\rangle \to |f\rangle\,\Big] = \frac{\langle\Omega_0|\,\mathscr{T}\Big\{\Big(\prod_{j=1}^{s}\frac{\alpha(\text{future},\mathbf{k}'_j)-\alpha(\text{past},\mathbf{k}'_j)}{\sqrt{2\omega(\mathbf{k}'_j)}}\Big)\Big(\prod_{j=1}^{r}\frac{\alpha^\dagger(\text{past},\mathbf{k}_j)-\alpha^\dagger(\text{future},\mathbf{k}_j)}{\sqrt{2\omega(\mathbf{k}_j)}}\Big)\Big\}\,|\Omega_0\rangle}{\langle\Omega_0|\,S\,|\Omega_0\rangle}$$

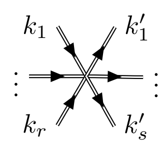
which, given our previous calculations, implies

$$\frac{\big\langle\mathbf{k}'_1\cdots\mathbf{k}'_s\big|\,S\,|\mathbf{k}_1\cdots\mathbf{k}_r\rangle}{\prod_{j=1}^{r}\Big(\sqrt{\frac{\pi}{\omega(\mathbf{k}_j)}}\Big)\prod_{j=1}^{s}\Big(\sqrt{\frac{\pi}{\omega(\mathbf{k}'_j)}}\Big)} = i^{r+s}\prod_{j=1}^{r}\big(k_j^2+m^2\big)\prod_{j=1}^{s}\big(k_j'^2+m^2\big)$$



This is known as the LSZ formula. For translationally invariant theories, the Fourier transforms are proportional to a delta function guaranteeing the conservation of momentum. It will prove convenient to define the matrix element $\mathcal{M}$ as the constant of proportionality

$$\frac{\big\langle\mathbf{k}'_1\cdots\mathbf{k}_s\big|\,S\,|\mathbf{k}_1\cdots\mathbf{k}_r\rangle}{\prod_{j=1}^{r}\Big(\sqrt{\frac{\pi}{\omega(\mathbf{k}_j)}}\Big)\prod_{j=1}^{s}\Big(\sqrt{\frac{\pi}{\omega(\mathbf{k}'_j)}}\Big)} = \mathcal{M}\Big(\;\cdots\;\Big)\delta(k_1+\cdots+k_r-k'_1-\cdots-k'_s)$$



For the $\lambda\phi^4$ theory, the matrix element corresponding to each Fourier space diagram is given by

$$\mathcal{M}(D) \equiv (-)^{\mathcal{E}_{ext.}}(2\pi)^{\frac{n+1}{2}(2\mathcal{V}-\mathcal{E})}\int(\text{unconstrained momenta})\prod_{e\in\mathcal{E}_{int.}}\tilde{D}(e)$$

11.2 Cross Sections

Now, we are in a situation to compute our first quantity that can be measured conveniently in the lab, namely the two-particle scattering differential cross sections. To do this, consider a two particle state describing a head-on collision with almost definite momenta.

$$|\psi_A\psi_B\rangle \equiv \int d\mathbf{k}_A d\mathbf{k}_B\,\psi_A(\mathbf{k}_A)\psi_B(\mathbf{k}_B)\,|\mathbf{k}_A\mathbf{k}_B\rangle$$

to consider a collision with non zero collision parameter $\mathbf{b}$, all we need to do is to multiply the wave function $\psi_B(\mathrm{k}_B)$ by $e^{-i\mathbf{b}.\mathbf{k}_B}$. From all this, we can write the differential cross section for the process

$$k_A, k_B \longrightarrow k'_1, \cdots, k'_s$$

as

$$\frac{d\sigma}{d\mathbf{k}'_1\cdots d\mathbf{k}'_s} = \int d^{n-1}\mathbf{b}\;\big|\,\big\langle\mathbf{k}'_1\cdots\mathbf{k}'_s\big|\,S\,|\psi_A\psi_B,\mathbf{b}\rangle\,\big|^2$$

$$= \int d^{n-1}\mathbf{b} \int d\mathbf{k}_A d\mathbf{k}_B d\mathbf{k}'_A d\mathbf{k}'_B \psi_A(\mathbf{k}_A)\psi_B(\mathbf{k}_B)\psi^*_A(\mathbf{k}'_A)\psi^*_B(\mathbf{k}'_B) e^{i\mathbf{b}.(\mathbf{k}'_B - \mathbf{k}_B)}$$

$$\left\langle \mathbf{k}'_1 \cdots \mathbf{k}'_s \right| S \left| \mathbf{k}_A \mathbf{k}_B \right\rangle \left\langle \mathbf{k}'_A \mathbf{k}'_B \right| S \left| \mathbf{k}'_1 \cdots \mathbf{k}'_s \right\rangle$$

$$= (2\pi)^{n-1} \int d\mathbf{k}_A d\mathbf{k}_B d\mathbf{k}'_A d\mathbf{k}'_B \psi_A(\mathbf{k}_A)\psi_B(\mathbf{k}_B)\psi^*_A(\mathbf{k}'_A)\psi^*_B(\mathbf{k}'_B)\delta_\perp(\mathbf{k}_B - \mathbf{k}'_B)\mathcal{M}\mathcal{M}'^*$$

$$\frac{\pi^s}{\prod_{j=1}^s \omega(\mathbf{k}'_j)} \frac{\pi^2}{\sqrt{\omega(\mathbf{k}_A)\omega(\mathbf{k}_B)\omega(\mathbf{k}'_A)\omega(\mathbf{k}'_B)}} \delta\Big(k_A + k_B - \sum_{j=1}^s k'_j\Big)\delta\Big(k'_A + k'_B - \sum_{j=1}^s k'_j\Big)$$

Where $\mathcal{M}$ and $\mathcal{M}'$ are the relevant matrix elements. Using the delta functions to perform the integrals over $\mathbf{k}'_A$ and $\mathbf{k}'_B$ we get

$$\boxed{\frac{d\sigma}{d\mathbf{k}'_1 \cdots d\mathbf{k}'_s} = (2\pi)^{n-1} \int \frac{d\mathbf{k}_A d\mathbf{k}_B}{|\beta_A - \beta_B|} |\psi_A(\mathbf{k}_A)|^2 |\psi_B(\mathbf{k}_B)|^2 |\mathcal{M}|^2 \delta\Big(k_A + k_B - \sum_{j=1}^s k'_j\Big) \frac{\pi^{2+s}}{\omega(\mathbf{k}_A)\omega(\mathbf{k}_B)\prod_{j=1}^s \omega(\mathbf{k}'_j)}}$$

At this stage, we are almost done in extracting a verifiable result from our theoretic calculations. For example in 3+1 dimensions the formula above may be used to find the two particle to two particle collision cross section in the CM frame as

$$\boxed{\left(\frac{d\sigma}{d\Omega}\right)_{CM} = \frac{2\pi^6}{E_{CM}^2}|\mathcal{M}|^2}$$

It must be added that the formula above has an extra factor of $4 = 2 \times 2$. This is to convert the Bosonic to Bosonic cross section area to the classical one. Up to the first order in λ, we have only one diagram to evaluate in the matrix element.

$$\mathcal{M}_{(1)} = -i\lambda(2\pi)^{-4}$$

which gives

$$\sigma = \frac{\lambda^2}{32\pi E_{CM}^2}$$

11.3 Decay Rates

The particles in the $\lambda\phi^4$ theory are stable since there are no kinematically allowed decay processes. Therefore, in this section, we use another toy model

$$\mathcal{L} = -\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{Z}{2}(\partial\chi)^2 - \frac{g}{2}\phi\chi^2$$

Now for an almost still ϕ-particle

$$|\psi\rangle = \int d\mathbf{k}\psi(\mathbf{k}) |\mathbf{k}_\phi\rangle$$

we want to know the transition probabilities

$$\mathbb{P}\Big[\, |\psi\rangle \longrightarrow |\mathbf{k}_{1,\chi}, \mathbf{k}_{2,\chi}, \cdots\rangle \Big]$$

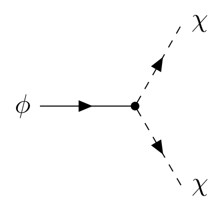
of course such a probability is proportional to the decay rate, Γ, as well as the waiting time: 'future − past'. The full decay rate is therefore given by

$$\frac{1}{\tau} = \Gamma = \lim_{\text{localized } \psi} \sum_{|f\rangle} \frac{|\langle f| S |\psi\rangle|^2}{\text{future} - \text{past}}$$

The first non-zero term for Γ in this theory will be given by

$$\Gamma^{(2)}_{\phi\to\chi+\chi} = \frac{\Omega_{n-1}\pi^2}{4m^2}\left(\frac{m}{2}\right)^{n-3}|\mathcal{M}|^2$$

where $\mathcal{M}$ is the matrix element corresponding to the diagram



The final expression for this quantity is found in the exercises.

The astute reader might have realised by now that higher order diagrams may give infinite results as physical quantities. Alleviating this apparent obstacle in a logically consistent way will be our main task in the next part.

Exercises

1. In $n+1$ space-time dimensions, check that the equations derived in this chapter are dimensionally consistent.

2. Evaluate the first matrix element corresponding to the decay process

$$\phi \to \chi + \chi$$

and find the first approximation for the half-life of such a particle.

Part IV

Renormalization

Chapter 12

Renormalizing the $\lambda\phi^4$ Theory

While evaluating more complex

Part V

Quantum Electrodynamics

Chapter 13

Free Fermions

Chapter 14

Free Photons

Chapter 15

The QED

Part VI

Conformal Field Theory

Chapter 16

Conformal Transformations

In this chapter we introduce the notion of conformal transformations. This will be essential for our discussion of field theories with corresponding symmetries. A conformal transformation on a manifold is one that does not change the angle between different vectors. From a passive point of view, this is a coordinate transformation $x^\mu \to x'^\mu$ such that

$$g'_{\mu\nu} = \Omega^2(x) g_{\mu\nu}$$

In this regard, the set of all possible conformal transformations on a manifold, $\mathcal{M}$, forms a group, $\texttt{conf}(\mathcal{M})$, generally known as the conformal group of the manifold.

To find the generators of the conformal group, we seek infinitesimal transformations $x^\mu \to x^\mu + \varepsilon\xi^\mu$ that are conformal. The condition on ξ^μ is called the conformal Killing equation

$$\pounds_\xi g_{\mu\nu} = \nabla_{\{\mu}\xi_{\nu\}} = \kappa g_{\mu\nu}$$

A vector field that satisfies this equation, is called a conformal Killing field and the scalar constant κ is called a conformal Killing factor. It is readily observed that the conformal Killing vector is given by

$$\kappa = \frac{2}{n}\nabla_\mu \xi^\mu$$

where n is the dimension of the manifold.

Our discussion in this chapter will be entirely dedicated to flat manifolds with the constant, flat metric tensor $\eta_{\mu\nu}$. Therefore it is useful to write the alleged conformal Killing field as

$$\xi_\mu = S^{(0)}_\mu + S^{(1)}_{\mu\alpha_1} x^{\alpha_1} + S^{(2)}_{\mu\alpha_1\alpha_2} x^{\alpha_1} x^{\alpha_2} + \cdots$$

$$+\Omega^{(1)}_{\mu\alpha_1} x^{\alpha_1} + \Omega^{(2)}_{\mu\{\alpha_1}\Sigma^{(1)}_{\alpha_2\}} x^{\alpha_1} x^{\alpha_2} + \Omega^{(3)}_{\mu\{\alpha_1}\Sigma^{(2)}_{\alpha_2\alpha_3\}} x^{\alpha_1} x^{\alpha_2} x^{\alpha_3} + \cdots$$

where S^i and Σ^i are completely symmetric tensors and Ω^i are anti-symmetric tensors of rank 2. On the other hand, the second order derivatives of the conformal Killing factor κ are highly restricted by the conformal Killing equation.

$$(n-2)\partial_\alpha\partial_\beta\kappa + g_{\alpha\beta}\partial^\mu\partial_\mu\kappa = g^{\mu\nu}\Big(g_{\alpha\beta}\partial_\mu\partial_\nu + g_{\mu\nu}\partial_\alpha\partial_\beta - g_{\mu\alpha}\partial_\nu\partial_\beta - g_{\nu\beta}\partial_\alpha\partial_\mu\Big)\kappa$$

$$= g^{\mu\nu}\Big[\big(\partial_\mu\partial_\nu\partial_\beta - \partial_\nu\partial_\beta\partial_\mu\big)\xi_\alpha + \big(\partial_\mu\partial_\nu\partial_\alpha - \partial_\alpha\partial_\mu\partial_\nu\big)\xi_\beta$$

$$+\big(\partial_\alpha\partial_\beta\partial_\nu - \partial_\nu\partial_\beta\partial_\alpha\big)\xi_\mu + \big(\partial_\alpha\partial_\beta\partial_\mu - \partial_\alpha\partial_\mu\partial_\beta\big)\xi_\nu\Big] = 0$$

Finally, taking a trace, we find the conditions

$$\Box\kappa = 0; \quad (n-2)\partial_\mu\partial_\nu\kappa = 0$$

Clearly, there is a stark difference here between $n = 2$ and $n > 2$ dimensions. We consider each case separately.

16.1 $n > 2$ dimensions

For $n > 2$, we have $\partial_\mu\partial_\nu\kappa = 0$ and therefore

$$\xi_\mu = t_\mu + \omega\eta_{\mu\nu}x^\nu + \Omega_{\mu\nu}x^\nu + \gamma_{\mu\alpha\beta}x^\alpha x^\beta$$

The first three terms correspond to translations, rotations and scale transformations respectively. But what transformation does the tensor γ_{abc} represent? The conformal Killing equation is equivalent to

$$\gamma_{\mu\alpha\beta} + \gamma_{\alpha\mu\beta} = \frac{2}{n}\eta_{\mu\alpha}\gamma^\nu{}_{\sigma\beta}$$

the solution is

$$\gamma_{\mu\alpha\beta} = \eta_{\mu\alpha}\sigma_\beta + \eta_{\mu\beta}\sigma_\alpha - \eta_{\alpha\beta}\sigma_\mu$$

Counting the number of generators, we find that there are $\frac{(n+1)(n+2)}{2}$ independent generators. In fact, it is possible to show that under the commutator operation

$$[A, B]^\mu \equiv B^\nu\partial_\nu A^\mu - A^\nu\partial_\nu B^\mu = \pounds_B A^\mu$$

these generators have the same Lie algebra as the group $SO(p+1, q+1)$ where (p, q) is the signature of the manifold.

16.2 $n = 2$ dimensions

To examine the $n = 2$ case, we start by formally allowing the real coordinates x^μ to take values in the complex plane. Although complex points do not exist on the manifold, we may always analytically the coordinate transformation rules to transformation rules between complex coordinates. As we'll see, this will help us express and classify the conformal transformations in well known terms.

Let us start our discussion with the case of the Euclidean plane; this is the non-compact set of real pairs of numbers (x, y), equipped with the metric

$$ds^2 = dx^2 + dy^2$$

If we define the *independent* complex coordinates

$$z \equiv x + iy; \quad \bar{z} \equiv x - iy$$

then the metric becomes

$$ds^2 = dz\,d\bar{z}.$$

Of course the true manifold points only correspond to coordinates satisfying $\bar{z} = z^*$. Nevertheless, any (extended) coordinate transformation is written as

$$(z, \bar{z}) \to (z', \bar{z}')$$

And therefore the metric transforms as

$$ds^2 = \Big(\frac{\partial z}{\partial z'}dz' + \frac{\partial z}{\partial \bar{z}'}d\bar{z}'\Big)\Big(\frac{\partial \bar{z}}{\partial z'}dz' + \frac{\partial \bar{z}}{\partial \bar{z}'}d\bar{z}'\Big) \overset{!}{=} \Omega^2(z', \bar{z}')dz'\,d\bar{z}'$$

Where the last equality holds, if and only if, the transformation is a conformal one. Clearly, there are only two possibilities here, either this is a holomorphic map, i.e.

$$z' = f(z), \quad \bar{z}' = \bar{f}(\bar{z})$$

or, an anti-holomorphic map, i.e.

$$z' = \bar{f}(\bar{z}), \quad \bar{z}' = f(z)$$

where f and $\bar{f}$ are independent, analytic (otherwise, the partial derivatives would be meaningless) functions. We still need to translate these transformations back into real coordinate systems; there, we need to have both $\bar{z} = z^*$ and $\bar{z}' = z'^*$ to hold. Therefore (dropping the redundant transformation rule for barred coordinates), the two holomorphic and anti-holomorphic cases correspond to

$$z' = f(z); \qquad z' = \Big(f(z)\Big)^*$$

for some analytic function f. We will generally be interested in the connected component of $\texttt{conf}(\mathbb{E}^2)$ that includes the identity, namely the holomorphic transformations.

Using the MacLaurin series for analytic functions, we may introduce the generators L_s that correspond to infinitesimal [1] transformations

$$\big(1 + \varepsilon L_s\big)(z) = z + \varepsilon z^{s+1}, \qquad s \in \mathbb{Z}.$$

Then, it is easy to see that these generators obey the Witt algebra under commutation operations.

$$\text{Witt algebra:} \qquad [L_s, L_r] = (s - r)L_{s+r}$$

Finally, let us add the important remark that although any analytic function is a good local conformal transformation, only the ones with exactly one pole and one zero in $\mathbb{C} \cup \{\infty\}$, namely the Møbius transformations are valid, global conformal transformations.

$$z' = \frac{az + b}{cz + d}; \qquad ad - bc = 1$$

In the exercises, the reader is further familiarized with this *global* subgroup. Note that this subgroup corresponds to translations, rotations, scale transformations and SCTs; just like in $n > 2$ dimensions.

For the 1+1 dimensional Minkowski space time, $\mathbb{E}^{1+1}$ with the metric

$$ds^2 = dx^2 - dt^2,$$

it is more convenient to use the pair of complex coordinates

$$z = x - t; \qquad \bar{z} = x + t.$$

[1] For the moment, we are focusing on a local domain in the complex plane where both $|z|$ and $|1/z|$ are bounded. We will deal with the *global* transformations in a moment.

once again, the metric becomes

$$ds^2 = dz\, d\bar{z}$$

And therefore, a similar piece of reasoning, allows us to write any conformal transformation in one of the following two forms

$$z' = f(z); \qquad \bar{z}' = \bar{f}(\bar{z})$$

$$z' = \bar{f}(\bar{z}); \qquad \bar{z}' = f(z).$$

Here, the actual manifold corresponds to $z, \bar{z} \in \mathbb{R}$ and therefore, a valid transformation corresponds to *real-entire* functions f, $\bar{f}$. Once again, we usually neglect the component that is not connected to the identity. What remains is

$$x - t \to f(x - t); \qquad x + t \to \bar{f}(x + t)$$

with monotonically increasing bijections $f, \bar{f} : \mathbb{R} \to \mathbb{R}$.

16.3 Exercises

1. On flat space-times, we proved that the conformal Killing factors satisfy

$$(n-2)\partial_\mu \partial_\nu \kappa + g_{\mu\nu} \Box \kappa = 0$$

a) First, show that on curved manifolds

$$(n-2)\nabla_\mu \nabla_\nu \kappa + g_{\mu\nu} \Box \kappa$$

$$= 2\nabla_\alpha \big(R^\alpha{}_{\nu\mu\beta} \xi^\beta\big) - 2\nabla_\mu \big(R_{\nu\alpha} \xi^\alpha\big) + R_{\mu\alpha}\big(\nabla^\alpha \xi_\nu - \nabla_\nu \xi^\alpha\big) + R^\alpha{}_{\mu\nu\beta}\big(\nabla_\alpha \xi^\beta - \nabla^\beta \xi_\alpha\big)$$

⚓ b) Simplify the RHS of the previous part as much as possible.

2. For $p + q > 2$, identify the Lie algebra for $\texttt{conf}(\mathbb{E}^{p+q})$ with that of the group $\texttt{SO}(p+1, q+1)$.

3. a) For the Euclidean manifold $\mathbb{E}^n$, show that the following discrete transformation is conformal

$$x'^\mu = \frac{x^\mu}{x^\nu x_\nu}$$

b) Find the conformal Killing vector field corresponding to the following, infinitesimal conformal transformation

$$x''^\mu = \frac{x'^\mu + \varepsilon t^\mu}{(x'^\nu + \varepsilon t^\nu)(x_{\prime\nu} + \varepsilon t_\nu)}$$

and show that this is a $\gamma-$transform.

4. a) Show that the Mobius transforms are generated using the Witt generators $\{L_{-1}, L_0, L_{+1}\}$.

⚓ b) Show that the Møbius subalgebra is the only non trivial (containing more than 1 and less than all of the generators) subalgebra of the Witt algebra.

c) Show that under the Møbius transforms, a line is mapped to either a circle or a line and that a circle is mapped to a line or a circle.

d) Let M_1 and M_2 be two Møbius trnasformations as

$$M_i(z) = \frac{a_i z + b_i}{c_i z + d_i}; \quad i = 1, 2$$

Show that the composite transform

$$M_3(z) \equiv M_2\big(M_1(z)\big) = \frac{a_3 z + b_3}{c_3 z + d_3}$$

satisfies

$$\begin{pmatrix} a_3 & b_3 \\ c_3 & d_3 \end{pmatrix} = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$$

Chapter 17

Conformal Fields

From now on and until the end of this part (while talking about CFTs), we call any physical quantity that depends on space-time positions, a field; examples include: the independent fields that appear in the Lagrangian, the Lagrangian itself, the energy-momentum tensor, etc. Of particular interest are the quasi-primary fields. A field Φ is called quasi-primary if, under the infinitesimal conformal transformation

$$x^\mu \to x'^\mu = x^\mu + \varepsilon \xi^\mu$$

it transforms as

$$\Phi \to \Phi' = \Phi - \varepsilon\Big(\pounds_\xi \Phi + \frac{1}{2}\Delta_\Phi \kappa \Phi\Big)$$

where Δ_Φ is a constant, called the scaling dimension of the field Φ. Under this, the Lagrangian changes as

$$\delta\mathcal{L} = -\varepsilon\frac{\kappa}{2}\Big[2g^{\mu\nu}\frac{\partial\mathcal{L}}{\partial g^{\mu\nu}} + \sum_a \Delta_a\big(\frac{\partial\mathcal{L}}{\partial\phi_a} + \frac{\partial\mathcal{L}}{\partial\nabla_\mu\phi_a}\nabla_\mu\phi_a\big) - n\mathcal{L}\Big]$$

Considering the Lagrangian as a sum of product terms

$$\mathcal{L} = \sum_i \mathcal{L}_i,$$

we get

$$\delta\mathcal{L} = -\varepsilon\frac{\kappa}{2}\sum_i \mathcal{L}_i\Big(2\#_g^i + \sum_a \Delta_a \#_a^i - n\Big)$$

where $\#_g^i$ and $\#_a^i$ denote the power of inverse metric tensors and ϕ_a in each term. As an example, consider the term

$$\phi^2(\nabla_\alpha\phi)g^{\alpha\beta}(\nabla_\beta\psi_{\mu\nu})g^{\mu\rho}g^{\nu\sigma}\chi_\rho\chi_\sigma$$

this will be conformally invariant, only if

$$6 + 3\Delta_\phi + \Delta_\psi + 2\Delta_\chi = n$$

Most of the time, when we talk about conformal Lagrangians, we are pointing to those that exhibit conformal symmetry with $\Delta_a = 0$. Each term in such a theory must have exactly $n/2$ inverse metric tensors involved. For instance in $3+1$ dimensions, the free electromagnetic theory would be conformal

$$\mathcal{L}_{EM} = -\frac{1}{4}g^{\mu\nu}g^{\alpha\beta}(\nabla_\alpha A_\mu - \nabla_\mu A_\alpha)(\nabla_\beta A_\nu - \nabla_\nu A_\beta)$$

An immediate consequence of such a definition, would be that the Hilbert tensor is traceless. The Hilbert tensor is dfeined as

$$H_{\mu\nu} \equiv -2\frac{\partial \mathcal{L}}{\partial g^{\mu\nu}} + g_{\mu\nu}\mathcal{L}$$

and therefore

$$\begin{aligned} H^{\mu}_{\mu} &= n\mathcal{L} - 2g^{\mu\nu}\frac{\partial \mathcal{L}}{\partial g^{\mu\nu}} \\ &= (n - 2\#_g)\mathcal{L} = 0 \end{aligned}$$

For many interesting cases, the Hilbert tensor is the same as the Belinfante energy momentum tensor; in those cases, the energy momentum tensor is traceless as well.

$$T^{\mu}_{Bel.\,\mu} = 0$$

17.1 Conformal Correlation Functions

In a conformal theory (more specifically, for a conformal state), we expect the quantum correlation functions to be conformally symmetric as well. This strongly limits the form of such functions.

Part VII

Quantum Field Theories on Curved Space-Times

Chapter 18

Static Space-times

In this chapter, we start by a simple generalisation and assume the space to have arbitrary, but time-invariant geometry. We also assume that the space-time is static, therefore

$$ds^2 = -dt^2 + h_{ij}dx^i dx^j$$

We also neglect the effect of the fields on space-time geometry for now.

To solve the problem of non-interacting fields, we used Fourier transforms. Here, and on a curved space, we can not do that any more and therefore seek alternatives. For two square integrable, complex scalar fields ϕ, ψ, we define the inner product

$$\langle \phi, \psi \rangle \equiv \int_\Sigma dx \sqrt{h} \phi^*(x) \psi(x)$$

Integration by parts reveals that for any real vector field V, the operator $V^i \nabla_i$ is anti-hermitian, that is

$$\langle \phi, V^i \nabla_i \psi \rangle = -\langle V^i \nabla_i \phi, \psi \rangle$$

However, the Laplacian operator $\nabla^2 \equiv h^{ij} \nabla_i \nabla_j$ is a negative definite operator. This allows us to define a Fourier-like transform with analysis and synthesis equations as

$$\tilde{\psi}(k, \Pi) \equiv \int dx \sqrt{h}\; \chi^*(k, \Pi; x) \psi(x)$$

$$\psi(x) = \sum\!\!\!\!\!\!\int_{k,\Pi} \chi(k, \Pi; x) \tilde{\psi}(k, \Pi)$$

where the functions $\chi(k, \Pi; x)$ are the properly normalized eigen-functions of the Laplacian operator with eigenvalue $\lambda = -k^2$ and degeneracy index Π.

18.1 The Klein-Gordon Fields

The Klein-Gordon Lagrangian on a curved space is

$$\mathcal{L} = -\frac{1}{2} \sum_a \left[g^{\mu\nu} (\nabla_\mu \phi_a)(\nabla_\nu \phi_a) + m^2 \phi_a^2 \right]$$

the conjugate momenta become

$$\pi_a(x) = \dot{\phi}_a(x)$$

and the Hamiltonian is

$$H = -\frac{1}{2}\sum_a \int_\Sigma dx\,\sqrt{h}\,\Big[\Big]$$

Part VIII

AdS/CFT

Part IX

Quantum Gravity

In this part, we will try to quantise Einstein's theory of general relativity. The spacetime dimension is set to $3+1$ as default; although some results will be generalised to $n+1$ dimensions.

Chapter 19

Quantum Geometry

Our attempt at quantizing the theory of gravity, which in turn is a theory of pseudo-Riemannian geometry, starts with quantizing area vectors. For a small plane with some area vector $\mathbf{A}$, we don't see conjugate coordinates to serve as the starting point of our canonical commutation relations. Therefore, we impose the first postulate, as the commutation relation for different components of the area vector

$$[A^i, A^j] = 2i\gamma\varepsilon^{ij}_{\ \ k}A^k$$

From here, we already see that the area of a plane has a discrete spectrum

$$\sqrt{|\mathbf{A}|^2} = 2\gamma\sqrt{j(j+1)}$$

For $j \gg 1$, this expression is approximated as

$$|\mathbf{A}| \sim \gamma(2j + 1 + \mathcal{O}(1/j))$$

19.1 The Tetrahedron

As a first example for a geometric quantum system, let us consider a tetrahedron with surface area vectors $\mathbf{A}_a$ satisfying

$$\sum_{a=1}^{4} \mathbf{A}_a = 0$$

The area vectors determine all geometric properties of the tetrahedron uniquely without having any extra constraints imposed on them. Specifically, the volume is given by

$$V^2 = \frac{2}{9}\det\big(\mathbf{A}_1\,\mathbf{A}_2\,\mathbf{A}_3\big)$$

Or, equivalently

$$V = \frac{\sqrt{2}}{3}\det\Big(\{\mathbf{A}_i.\mathbf{A}_j\}_{i,j=1}^{3}\Big)^{1/4}$$

Although the dot products do not commute, we know their spectrum individually due to

$$\mathbf{A}.\mathbf{A}' = \frac{1}{2}\left[|\mathbf{A}+\mathbf{A}'|^2 - A^2 - A'^2\right]$$

we may use this to find a semi-classical approximation for the volume spectrum of a tetrahedron. This is done in the exercises.

19.2 Maximal Tesselation of $\mathbb{S}^2$

The Hilbert space of a quantum polyhedra is the spin-zero (or the invariant) subspace of the spin sum space where different area operators add up to zero. For N faces, this is written as

$$\mathcal{H} = \bigoplus_{\frac{1}{2} \leq j_1 \leq \cdots \leq j_N} \mathrm{Inv}_{SU(2)} \left[\mathcal{H}_{j_1} \otimes \cdots \otimes \mathcal{H}_{j_N}\right]$$

In this regard, a maximally fine quantum tesselation of a sphere with area $A \sim \sqrt{3} N \gamma$ in the large N limit is described by the spin-zero subspace of $2N$ different spin-1/2 area vectors. The dimension of this subspace is

$$\dim \left[\mathrm{Inv}_{SU(2)} \left(\mathcal{H}_{1/2}^{\otimes 2N}\right)\right] = \frac{1}{N+1} \binom{2N}{N} \sim N^{-3/2} 4^N$$

to be proven in the exercises. Let Π be a pairing of the spins:

$$\Pi = \{\{\Pi_{1,0}, \Pi_{1,1}\}, \{\Pi_{2,0}, \Pi_{2,1}\}, \cdots, \{\Pi_{N,0}, \Pi_{N,1}\}\}$$

made of an unordered tuple of unordered pairs. The indices cover the set $[2N]$. Now define the $SU(2)$ invariant ket

$$|\Pi\rangle \equiv \bigotimes_{i=1}^{N} \left(\frac{\left|0_{\Pi_{i,0}} 1_{\Pi_{i,1}}\right\rangle - \left|1_{\Pi_{i,0}} 0_{\Pi_{i,1}}\right\rangle}{\sqrt{2}} \right)$$

One can show that these states provide an overcomplete basis for the invariant subspace (See Exercises).

19.2.1 Classical Description

Now let us focus on the scenario under which the number of faces $2N$ is taken to be very large, allowing for the possibility of quantum states that closely model a smooth convex body. Classically, to describe this body, it suffices to have $2N$ different unit vectors on the sphere $\{\hat{\mathbf{n}}_i\}_{i=1}^{2N}$. In the large N limit, this in turn can be described by a density function on the surface of $\mathbb{S}^2$, denoted by $\rho(\Omega)$. It satisfies

$$\int d\Omega \rho(\Omega) = 2N = \frac{A}{a_0}$$

$$\int d\Omega \rho(\Omega) \hat{\mathbf{n}}(\Omega) = \mathbf{0} \quad \text{The Closure Constraint}$$

Leading to

$$\rho(\Omega) = \frac{N}{2\pi} + \sum_{l \geq 2} \sum_{m=-l}^{+l} r_{lm} Y_{lm}(\Omega)$$

Since the $\{|\Pi\rangle\}$ are a basis for the Hilbert space, it makes sense to spend some time on their classical counter parts where $\{\hat{\mathbf{n}}_i\}_{i=1}^{N}$ are i.i.d. and isotropic. Then, the density ρ is given by

$$\rho(\Omega) = \sum_i \left[\delta(\Omega - \hat{\mathbf{n}}_i) + \delta(\Omega + \hat{\mathbf{n}}_i)\right]$$

All of the r_{lm} in this setting are large sums of i.i.d. variates which means we may invoke the central limit theorem.

$$r_{lm} = \sum_{i=1}^{N} \int d\Omega Y_{lm}^*(\Omega) [\delta(\Omega - \hat{\mathbf{n}}_i) + \delta(\Omega + \hat{\mathbf{n}}_i)] = \begin{cases} 2 \sum_{i=1}^{N} Y_{lm}^*(\hat{\mathbf{n}}_i) & l \text{ is even} \\ 0 & \text{otherwise} \end{cases}$$

The covariances are found to be

$$\langle r^*_{lm} r_{l'm'} \rangle = \frac{N}{\pi} \delta_{ll'} \delta_{mm'}$$

$$\langle r_{lm} r_{l'm'} \rangle = \frac{N}{\pi} (-)^m \delta_{ll'} \delta_{m,-m'}$$

This reveals the r_{lm} as Gaussian variables written in terms of standard i.i.d. variables X_{lm} and Y_{lm} (with $m > 0$) as

A two-fold pairing of the vectors in the form $\{(\hat{\mathbf{n}}, -\hat{\mathbf{n}})_i\}$ kills all the odd ℓ modes of $\rho(\Omega)$. However, our rotation invariance constraint is to only have the $l = 1$ modes suppressed. A slightly better approximation therefore would be to have a quad pairing of the unit vectors $\{(\hat{\mathbf{n}}_1, \hat{\mathbf{n}}_2, \hat{\mathbf{n}}_3, \hat{\mathbf{n}}_4)_i\}$ where for each pair the unit vectors add up to zero. One such four-pairing can be parametrized as a random rotation times

$$\hat{\mathbf{n}}_1 = \cos\frac{\theta}{2}\hat{\mathbf{x}} + \sin\frac{\theta}{2}\hat{\mathbf{y}}$$

$$\hat{\mathbf{n}}_2 = \cos\frac{\theta}{2}\hat{\mathbf{x}} - \sin\frac{\theta}{2}\hat{\mathbf{y}}$$

$$\hat{\mathbf{n}}_3 = -\cos\frac{\theta}{2}\hat{\mathbf{x}} - \sin\frac{\theta}{2}\left(\cos\psi\hat{\mathbf{y}} + \sin\psi\hat{\mathbf{z}}\right)$$

$$\hat{\mathbf{n}}_4 = -\cos\frac{\theta}{2}\hat{\mathbf{x}} + \sin\frac{\theta}{2}\left(\cos\psi\hat{\mathbf{y}} + \sin\psi\hat{\mathbf{z}}\right)$$

In general, all of these will lead to Gaussian fluctuations with spectra $r_{lm} \sim \sqrt{N}$.

The density function can be written in terms of the extrinsic curvature as

$$\rho(\Omega) = \frac{1}{a_0 \det(K)}$$

19.2.2 Quantum Description

First, one may ask for a quantum description of ρ. We have

$$(\frac{2\gamma}{a_0})^r \sum_{\alpha=1}^{2N} (\hat{\mathbf{m}}.\boldsymbol{\sigma}_\alpha)^r = \int d\Omega\, \rho(\Omega)\ [\hat{\mathbf{m}}.\hat{\mathbf{n}}(\Omega)]^r$$

Interestingly, all the odd powers vanish and even powers are also easy to compute.

Now let's see if there is a connection between fluctuations in ρ and fluctuations in the height of the convex object.

19.3 Exercises

1. Prove the following singlet formula

$$|l+l=0\rangle = \frac{1}{\sqrt{2l+1}} \sum_{m=-l}^{m=+l} (-)^m \, |m,-m\rangle$$

2. a) Prove the following more explicit volume formula for a tetrahedron

$$\frac{81}{4}V^4 = A_1^2 A_2^2 A_3^2 - A_1^2(\mathbf{A}_2.\mathbf{A}_3)^2 - A_2^2(\mathbf{A}_1.\mathbf{A}_3)^2 - A_3^2(\mathbf{A}_1.\mathbf{A}_2)^2 + 2(\mathbf{A}_2.\mathbf{A}_3)(\mathbf{A}_1.\mathbf{A}_3)(\mathbf{A}_1.\mathbf{A}_2)$$

   b) Furhter, show that

$$[(\mathbf{A}_1.\mathbf{A}_2), (\mathbf{A}_1.\mathbf{A}_3)] = 9i\gamma V^2$$

3. Write a code that finds the volume spectrum for a tetrahedron with $A_1 = A_2 = A_3 \leq 2\gamma j_{\max}(j_{\max}+1)$

4. Prove the equation

$$\dim\left[\mathrm{Inv}_{SU(2)}\left(\mathcal{H}_{1/2}^{\otimes 2N}\right)\right] = \frac{1}{N+1}\binom{2N}{N} \sim N^{-3/2}4^N$$

5. Show that

$$\mathrm{Span}(\{|\Pi\rangle\}) = \mathrm{Inv}_{SU(2)}\left(\mathcal{H}_{1/2}^{\otimes 2N}\right)$$

6. Let y^a be a coordinate system for the manifold describing K different unit vectors that add up to zero quotient w.r.t rotations. Define the matrix

$$N(y^a) \equiv [\hat{\mathbf{n}}_1 \cdots \hat{\mathbf{n}}_K]$$

   Show that the measure

$$\left(\prod_{i=1}^{K} d\Omega_i\right)\delta(\hat{\mathbf{n}}_1 + \cdots + \hat{\mathbf{n}}_K) \propto (\prod_a dy)\sqrt{|h|}\sqrt{|K - N(y^a)N^T(y^a)|}$$

   where h_{ab} is the induced metric

$$h_{ab} = \sum_{i=1}^{K} \frac{\partial \hat{\mathbf{n}}_i}{\partial y^a}.\frac{\partial \hat{\mathbf{n}}_i}{\partial y^b}$$

Part X

Appendices

Appendix A

The Feynman Propagator in Space-time

For spacelike vectors, we use a Lorentz transformation to set $t = 0$, then the Feynman propagator in a distance r becomes

$$D_F(0, r\hat{\mathbf{z}}) = \frac{1}{\Gamma\left(\frac{n-1}{2}\right)\sqrt{\pi}(4\pi)^{n/2}} \int_m^\infty d\omega\, (\omega^2 - m^2)^{n/2-1} \int_0^\pi d\theta \cos\left(\sqrt{\omega^2 - m^2} r \cos\theta\right) \sin^{n-2}\theta$$

and for timelike separations, setting $\mathbf{x} = 0$, we get

$$D_F(t, \mathbf{0}) = \frac{1}{\Gamma\left(\frac{n}{2}\right)(4\pi)^{n/2}} \int_m^\infty d\omega\, (\omega^2 - m^2)^{n/2-1} \cos\left(\omega t\right)$$

Although these integrals may diverge in some dimensions, we do not worry too much about it. Such divergences are equivalent to those appearing in a stochastic setting where one considers a white noise; the correlation function may be diverging but at the end of the day, the interesting quantities will turn out to be finite.

Specifically for $n = 3$ these become

$$D_F(0, r\hat{\mathbf{z}}) = \frac{1}{4\pi r} \int_m^\infty d\omega\, \sin\left(r\sqrt{\omega^2 - m^2}\right)$$

$$D_F(t, \mathbf{0}) = \frac{1}{4\pi^2} \int_m^\infty d\omega\, \cos\left(\omega t\right) \sqrt{\omega^2 - m^2}$$

Appendix B

Lie Groups

Consider a smooth continuous matrix group. The tangent space at the identity, $\mathbb{T}_1$, includes linearly independent matrices G_i called the generators of the group. For two such matrices G_1, G_2 and two group members

$$g_1 = 1 + \varepsilon G_1 + \varepsilon^2 A_1 + \cdots$$

$$g_2 = 1 + \varepsilon G_2 + \varepsilon^2 A_2 + \cdots$$

the group commutator yields

$$g_1 g_2 g_1^{-1} g_2^{-1} = 1 + \varepsilon^2 [G_1, G_2] + \cdots$$

In order for both sides to belong to the group, we need

$$[G_1, G_2] \in \mathbb{T}_1$$

In general, this implies the following relations

$$[G_j, G_k] = C^i_{jk} G_i$$

for some array of numbers C^i_{jk}. This is a Lie algebra on the tangent space and is called the Lie algebra of the group. If the group is denoted by some symbol G, then its Lie algebra is denoted by the Fraktur symbol $\mathfrak{g}$.

All members of the (identity-connected component of the) Lie group are written as exponentials of the generators, that is

$$\exp\left(\theta_i G_i\right)$$

In this sense, the θ_i form a coordinate system for the group; in this coordinate system, the multiplication laws are determined from the Lie algebra

$$\left[\underset{\sim}{\theta}\right] \times \left[\underset{\sim}{\theta'}\right] = \left[\underset{\sim}{\theta''}\right]$$

However, the specific form of the generator matrices may further simplify the multiplication law: the matrix logarithm is not unique and therefore, it might happen that different coordinate values point to the same group element. For this reason, two groups with similar Lie algebras can be different (i.e. not be isomorphic).

As an example, take the 2D translation group $T(2)$. This may be generated by

$$G_1 = \begin{pmatrix} 1 & \\ & \end{pmatrix} , \; G_2 = \begin{pmatrix} & \\ & 1 \end{pmatrix}$$

The Lie algebra is

$$\mathfrak{t}(2): \quad [G_1, G_2] = 0$$

this is a non-compact group with a one to one correspondence between the θ coordinates and the group elements. On the other hand, one can represent the compact unitary group of $U(1) \otimes U(1)$ with the same Lie algebra

$$G_1 = \begin{pmatrix} i & \\ & \end{pmatrix} , \; G_2 = \begin{pmatrix} & \\ & i \end{pmatrix}$$

$$[G_1, G_2] = 0$$

This similarity of the Lie algebras

$$\mathfrak{t}(2) = \mathfrak{u}(1) \otimes \mathfrak{u}(1)$$

does not imply the isomorphism

$$\text{WRONG:} \quad T(2) \cong U(1) \otimes U(1)$$

On the group manifold, we may define the metric

$$ds^2(\delta g, \delta g) = \mathrm{Tr}\left\{[(g + \delta g)g^{-1} - 1]^\dagger [(g + \delta g)g^{-1} - 1]\right\}$$

For unitary groups, this further simplifies into

$$ds^2 = \mathrm{Tr}\big(\delta g^\dagger \delta g\big)$$

Interestingly, left or right multiplicative actions are isometries of this metric. For a compact unitary group, the Haar measure is defined as the unique probability measure over the group that is invariant under left or right group action. The invariance of our metric over the group action means that the imposed integral measure ($\sqrt{g}$) is proportional to the Haar measure. This provides a computational route to finding the Haar measure for any unitary group.

Given the Haar measure, one can make a natural Hilbert space of complex square-integrable functions over the group manifold $\mathcal{G}$, sylmbolized by $L^2[\mathcal{G}]$. The dot product will be

$$\langle \phi, \psi \rangle = \int d\mu(g)\; \phi^*(g)\psi(g)$$

Acting on this Hilbert space, we may define the left and right derivative operators for each generator in the tangent space. This is how they act:

$$\partial^L_G \psi(v) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left[\psi(e^{\varepsilon G} v) - \psi(v)\right]$$

$$\partial^R_G \psi(v) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left[\psi(v e^{\varepsilon G}) - \psi(v)\right]$$

B.1 The $SU(2)$

The Lie group $SU(2)$ will play a central role in quantum mechanics and specially in the theory of quantum gravity. This is the set of special (unit determinant) 2×2 unitary matrices. Any member $v \in SU(2)$ can be parametrized as

$$v = \begin{pmatrix} t + iz & y + ix \\ -y + ix & t - iz \end{pmatrix}$$

with the condition $t^2 + x^2 + y^2 + z^2 = 1$. The manifold on which the members of the group $SU(2)$ lie, is therefore a 3-sphere $\mathbb{S}^3$. To make more sense of this, we may re-write any member as

$$v = e^{i(\psi/2)\hat{\mathbf{n}}.\boldsymbol{\sigma}} = \cos\frac{\psi}{2}\mathbb{I} + i \sin\frac{\psi}{2}\hat{\mathbf{n}}.\boldsymbol{\sigma}$$

where $\hat{\mathbf{n}}(\theta,\varphi)$ is the 3D unit vector parametrized in the polar coordinates (θ,φ) and the $\sigma_{1,2,3}$ are the Pauli matrices.

$$\sigma_1 = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} & -i \\ i & \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$$

In the (ψ,θ,φ) coordinates, the group metric will become

$$ds^2 = 2\left[d(\frac{\psi}{2})^2 + \sin^2\frac{\psi}{2}d\theta^2 + \sin^2\frac{\psi}{2}\sin^2\theta\, d\varphi^2\right] = 2d\Omega_3^2$$

The north pole ($\psi = 0$) in this coordinate system is the identity matrix, and the south pole $\psi = 2\pi$ is $-\mathbb{I}$. The Haar measure becomes

$$d\mu = \frac{d\Omega_3}{2\pi^2} = \frac{1}{4\pi^2}\sin^2(\frac{\psi}{2})\sin\theta\, d\psi\, d\theta\, d\varphi$$

Finally, a near identity, first order analysis, yields the generators of $SU(2)$ as

$$G_i = -\frac{i}{2}\sigma_i$$

the Lie algebra is therefore

$$\mathfrak{su}(2):\ [G_i, G_j] = \varepsilon_{ijk}G_k$$

B.1.1 Unitary Representations of $SU(2)$

For a generic unitary representation of the $SU(2)$ group, the generators G_i are anti-Hermitian and therefore the operators $J_i \equiv iG_i$ satisfying

$$[J_i, J_j] = i\varepsilon_{ijk}J_k$$

are Hermitian. Then the angular momentum theory determines the representations to be spinors with total spin number $j = 0, \frac{1}{2}, 1, \cdots$. The explicit unitary matrix representations are found by calculating the so called Wigner D matrices

$$D^j_{mn}(v) = \text{The } mn \text{ entry of the spin}-j \text{ unitary representation of the } SU(2) \text{ member } v$$

Setting up a locally flat coordinate system around each point, we find that the Wigner matrices satisfy

$$\left[\nabla_v^2 + \frac{1}{2}j(j+1)\right]D^j_{mn}(v) = 0$$

$$-i\partial^L_{L_3}D^j_{mn}(v) = mD^j_{mn}(v)$$

$$-i\partial^R_{L_3}D^j_{mn}(v) = nD^j_{mn}(v)$$

These are enough to prove the orthogonality of these functions. Now to normalize them, we consider the channel

$$\mathcal{E}(\rho) = \int d\mu(v)\, D(v)\rho D^\dagger(v)$$

The unknown that we're after are

$$\langle|D^j_{mn}(v)|^2\rangle$$

The channel $\mathcal{E}$ kills all the coherences and is therefore presented by a symmetric bistochastic matrix $B_{mn} = \langle|D^j_{mn}(v)|^2\rangle$. Finally, we may use the fact that $\mathcal{E}^2 = \mathcal{E}$ as well as the Perron-Frobenius theorem to find

$$\langle|D^j_{mn}(v)|^2\rangle = \frac{1}{2j+1}$$

In fact, these functions form a complete basis for the Hilbert space and we may write

B.2 Exercises

1. Show that the group $SU(2)$ is a double cover of the $SO(3)$ group via the two to one even map

$$R_{ij} = \delta_{ij}(1 - 2x^2) + 2x_i x_j - 2\varepsilon_{ijk} t x_k$$

where

$$t = \mathrm{Tr}\,\frac{v}{2} \quad ; \quad x_i = \frac{1}{2i}\,\mathrm{Tr}(v\sigma_i)$$

2. a) Show that $J^2 \equiv J_1^2 + J_2^2 + J_3^2$ commutes with all the generators of a $SU(2)$ representation. This means that for an irreducible representation, J^2 is proportional to identity.

   b) Using the $J_\pm = J_x \pm iJ_y$ ladder operators, show that the representations are indexed by the half-integer total angular momentum: $J^2 = j(j+1)\mathbb{I}$

3. Complete the proof sketch in the chapter to show that

$$\langle |D^j_{mn}(v)|^2 \rangle = \frac{1}{2j+1}$$

4. ⚓ Show that the Wigner-D matrices form a complete basis for the Hilbert space $L^2[SU(2)]$.

Appendix C

Symplectic Geometry

C.1 Symplectic and Contact Manifolds

Definition 1. On a vector space $\mathbb{V}$, a non-degenerate (full rank), bilinear, skew-symmetric map

$$\Omega : \mathbb{V} \times \mathbb{V} \to \mathbb{R}$$

is called a symplectic form.

Lemma 1. *Let Ω be a symplectic form on a vector space $\mathbb{V}$ with field $\mathbb{R}$. Then, there exists a basis, in which*

$$\Omega(\mathbf{v}_1, \mathbf{v}_2) = \mathbf{v}_1^T \begin{pmatrix} 0 & \mathbb{I}_{n\times n} \\ -\mathbb{I}_{n\times n} & 0 \end{pmatrix} \mathbf{v}_2$$

Proof. First, use an orthogonal transformation to diagonalize the arbitrary matrix form of Ω into blocks of the form

$$\begin{pmatrix} i\omega & \\ & -i\omega \end{pmatrix}$$

then use another even-odd change of basis to turn these into blocks of the form

$$\begin{pmatrix} & \omega \\ -\omega & \end{pmatrix}$$

So far, the basis change coefficients are real by virtue of symmetries. Finally, do a scale and shuffle to get to the desired matrix form.

□

Definition 2. A subspace $\mathbb{W} \subseteq \mathbb{V}$ is called symplectic if the restriction of Ω on it is non-degenerate. It is called isotropic if the restriction is null.

Definition 3. An exact two form $\Omega = d\omega$ is called symplectic if it has full linear algebraic rank as a matrix.

Clearly, such a form can only exist in manifolds with even dimensionality.

Definition 4. For a scalar field ϕ, and a symplectic differential ω, we define the symplectic gradient vector D^μ_ϕ as the unique solution to

$$\Omega_{\alpha\mu} D^\alpha_\phi = \partial_\mu \phi$$

Theorem 3. *Flow along a symplectic gradient, preserves the symplectic form; i.e.*

$$\pounds_{D_\phi} \Omega = 0$$

Proof. Let us just write this down

$$\pounds_{D_\phi} \Omega_{\alpha\beta} = D^\mu_\phi \partial_\mu \Omega_{\alpha\beta} + \Omega_{\mu\beta} \partial_\alpha D^\mu_\phi + \Omega_{\alpha\mu} \partial_\beta D^\mu_\phi$$

$$= D^\mu_\phi \big(\partial_\mu \Omega_{\alpha\beta} + \partial_\alpha \Omega_{\beta\mu} + \partial_\beta \Omega_{\mu\alpha}\big) + \partial_\alpha (\Omega_\mu \beta D^\mu_\phi) - \partial_\beta (\Omega_{\mu\alpha} D^\mu_\phi) = 0$$

□

Definition 5. A 1-form ω on an orientable, $2n-1$ dimensional manifold is called a contact form if $\omega \wedge d\omega^{\wedge(n-1)}$ is non-vanishing on the manifold.

Another, very important example of a symplectic manifold is the cotangent bundle $T^*\mathcal{M}$ of any manifold $\mathcal{M}$. There is a canonical symplectic form on a cotangent bundle, namely

$$\Omega = dx^\mu \wedge dp_\mu$$

this is the exterior derivative of the so called, tautological form

$$\Omega = -d\alpha \qquad \alpha \equiv p_\mu dx^\mu$$

It is clear that the formula for α is manifestly coordinate independent.

Definition 6. A diffeomorphism, Φ between two symplectic manifolds, $\mathcal{M}$ and $\mathcal{M}'$ is called a symplectomorphism between $(\mathcal{M}, \Omega)$, and $(\mathcal{M}', \Omega')$ if Ω' is the image of Ω under the diffeomorphism Φ.

Theorem 4. (Darboux) *All symplectic manifolds of the same dimensionality are locally symplectomorphic.*

Corollary 1. This means that for any symplectic manifold, there is a chart (called the Darbox chart) over which the symplectic form takes becomes

$$\Omega = \sum_i dx^i \wedge dy^i$$

Definition 7. The interior product of a $(k+1)-$form Ω and a vector A $\iota_A \Omega \equiv$, is the $k-$ forms that satisfies

$$(\iota_A \Omega)(V_1, \cdots, V_k) = \Omega(A, V_1, \cdots, V_k)$$

Theorem 5. **(Cartan's Identity)**

$$\pounds_A \Omega = d\iota_A \Omega + \iota_A d\Omega$$

Proof. For an $n-$form Ω we have

$$\begin{aligned}(d\Omega)_{\alpha\mu_1\cdots\mu_n} &= \frac{1}{n!}\partial_{[\alpha}\Omega_{\mu_1\cdots\mu_n]} \\ &= \partial_\alpha \Omega_{\mu_1\cdots\mu_n} - \sum_{r=1}^{n} \partial_{\mu_r}\Omega_{\mu_1\cdots\mu_{r-1}\alpha\mu_{r+1}\cdots\mu_n}\end{aligned}$$

therefore

$$(\iota_A d\Omega)_{\mu_1\cdots\mu_n} = A^\alpha\Big(\partial_\alpha \Omega_{\mu_1\cdots\mu_n} - \sum_{r=1}^{n} \partial_{\mu_r}\Omega_{\mu_1\cdots\mu_{r-1}\alpha\mu_{r+1}\cdots\mu_n}\Big)$$

On the other hand,

$$\begin{aligned}(d\iota_A\Omega)_{\mu_1\cdots\mu_n} &= \frac{1}{(n-1)!}\Big[(\partial_{\mu_1}A^\alpha)\Omega_{\alpha\mu_2\cdots\mu_n} + A^\alpha\partial_{\mu_1}\Omega_{\alpha\mu_2\cdots\mu_n}\Big]\Big|_{[\mu_1\cdots\mu_n]} \\ &= \sum_{r=1}^{n}\Big[(\partial_{\mu_r}A^\alpha)\Omega_{\mu_1\cdots\mu_{r-1}\alpha\mu_{r+1}\cdots\mu_n} + A^\alpha\partial_{\mu_r}\Omega_{\mu_1\cdots\mu_{r-1}\alpha\mu_{r+1}\cdots\mu_n}\Big]\end{aligned}$$

Adding the two expressions up, we get the definition of a Lie derivative.

□

C.2 Poisson Brackets

Definition 8. Let $(\mathcal{M}, \Omega)$ be a symplectic manifold. A vector field V is called symplectic if it preserves the symplectic form, i.e.

$$\pounds_V \Omega = 0$$

For such a vector field, we have

$$d\iota_V\Omega = d\iota_V\Omega + \iota_V d\Omega = \pounds_V\Omega = 0$$

Therefore all symplectic vector fields are (at least locally), a symplectic gradient. The difference between symplectic vector fields and symplectic gradients is determined by the topology of the manifold.

Lemma 2. *Let U and V be symplectic vector fields. Then,*

$$[U, V] = D_{\Omega(V,U)}$$

Proof.

$$\begin{aligned}\iota_{[U,V]}\Omega &= \pounds_U\iota_V\Omega - \iota_V\pounds_U\Omega \\ &= d\,\iota_U\iota_V\Omega = d\Omega(V,U)\end{aligned}$$

□

This means that we have a real Lie algebra over all symplectic or all Hamilton vector fields.

Definition 9. The Poisson bracket of two scalar fields on a symplectic manifold $(\mathcal{M}, \Omega)$ is defined as

$$\{\phi, \psi\} \equiv \Omega(D_\phi, D_\psi) = (d\phi)(D_\psi)$$

Lemma 3. *Here are a few properties of the Poisson brackets*

i) $D_{\{\phi,\psi\}} = [D_\psi, D_\phi]$

ii) Jacobi identity: $\{\phi, \{\psi, \chi\}\} + \{\psi, \{\chi, \phi\}\} + \{\chi, \{\phi, \psi\}\} = 0$

iii) Leibniz property: $\{\phi, \psi\chi\} = \{\phi, \psi\}\chi + \psi\{\phi, \chi\}$

C.3 Hamiltonian Dynamics

Definition 10. Let $(\mathcal{M}, \Omega)$ be a symplectic manifold. A submanifold $\mathcal{N} \subset \mathcal{M}$ is called a Lagrangian submanifold if at any point $P \in \mathcal{N}$, the projection of Ω onto $T_P\mathcal{N}$ vanishes. Clearly, for all such manifolds, we have

$$2 \dim \mathcal{N} \leq \dim \mathcal{M}$$

Definition 11. A Hamiltonian system $(\mathcal{M}, \Omega, H)$, with $\dim \mathcal{M} = 2n$, is called completely integrable if there are n independent functions $(F_1 = H, F_2, \cdots, F_n)$ such that

$$\{F_i, F_j\} = 0$$

Lemma 4. *For a fully integrable system, the level set of the functions $\{F_i\}$ form a Lagrangian submanifold.*

Theorem 6. *Let $(\mathcal{M}, \Omega, H, \{F_i\})$ be a completely integrable Hamniltonian system. The topology of each connected component of the level set for $\{F_i\}$ is of the form*

$$\mathbb{R}^{n-k} \times \mathbb{T}^k$$

for some $k \in [n]$.

Theorem 7. (Liouville-Arnold) *Let $(\mathcal{M}, \Omega, H, \{F_i\})$ be a fully integrable Hamiltonian system. Then, there exist a coordinate system $\{\theta_i\}$ on the Lagrangian submanifold of constant $\{F_i\}$ such that the dynamics becomes*

$$\dot{\theta}_i = const.$$

These coordinates are called the angle variables or angle coordinates.

Theorem 8. *It is locally possible to form a Darboux chart* $(\theta_1, \cdots, \theta_n; J_1, \cdots, J_n)$ *so that*

$$\{\theta_i, \theta_j\} = \{J_i, J_j\} = 0 \quad ; \quad \{\theta_i, J_j\} = \delta_{ij}$$

Definition 12. Let $(\mathcal{M}, \Omega)$ be a symplectic manifold and G a Lie group. Let $\psi : G \to \mathrm{Sympl}(G)$ a be a symplectomorphic action; i.e. a homomorphism. The action ψ is called a Hamiltonian action if

i)

$$\exists\, \mu : \mathcal{M} \to \mathfrak{g}^* \;\; s.t. \;\; \forall X \in \mathfrak{g} \; \iota_{\xi_X}\Omega = d\langle \mu(P), X\rangle$$

where ξ_X is the vector field corresponding to the generator $X \in \mathfrak{g}$, and $\langle ., .\rangle : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$ is the evaluation product.

ii)

$$\mu\Big(\psi_g(P)\Big) = \mathrm{Ad}^*_g \Big(\mu(P)\Big)$$

equivalent to

$$\forall\, X, Y \in \mathfrak{g} \;\; H_{[X,Y]} = \{H_X, H_Y\}$$

where

$$H_X(P) \equiv \langle \mu(P), X\rangle \,.$$

The map μ is then called the moment map.

C.4 The Tangent Space

The geodesic dynamics exists on the tangent bundle manifold $T\mathcal{M}$. The tangent space of this tangent bundle consists of vectors tangent to paths

$$\sigma(t) = \big[x^\mu(t)\,,\, v^\mu(t)\big]$$

We call one such vector *vertical* if it is tangent to a path where $\frac{dx^\mu}{dt} = 0$. Similarly, a vector is called *horizontal* if it is not vertical, and is tangent to a path where

$$\frac{dx^\alpha}{dt}\nabla_\alpha v^\mu = 0$$

In general, we found the tangent space of the tangent bundle is a direct sum

$$A = \frac{d\sigma}{dt} = (A_h, A_v) = \left(\frac{dx^\mu}{dt}, \frac{dx^\alpha}{dt}\nabla_\alpha v^\mu\right)$$

The Sasaki metric on the tangent bundle is defined as

$$\langle\langle A, B\rangle\rangle = \langle A_h, B_h\rangle + \langle A_v, B_v\rangle$$

where $\langle ., .\rangle$ is the metric on $\mathcal{M}$.

Define the linear operator $J : T_P T\mathcal{M} \to T_P T\mathcal{M}$ as

$$J(A_h, A_v) = (-A_v, A_h)$$

Using this we can define the symplectic form

$$\Omega(A, B) = \langle\langle JA, B\rangle\rangle = -\langle A_v, B_h\rangle + \langle A_h, B_v\rangle$$

C.5 Exercises

1. Show that $\mathbb{S}^2$ with $\Omega = d\big(\cos\theta\, d\varphi\big)$ is symplectic.

2. Show that
$$\iota_{[A,B]} = \big[\pounds_A, \iota_B\big]$$
Note that as a corollary, this implies
$$[\iota_A, \pounds_A] = 0$$

3. Show that on a cotangent bundle, and with a Hamiltonian scalar $H = H(x^\mu, p_\mu)$, the symplectic gradient flow is equivalent to
$$\dot{x}^\mu = \frac{\partial H}{\partial p_\mu} \quad ; \quad \dot{p}_\mu = -\frac{\partial H}{\partial x^\mu}$$

Appendix D

Paths' Math

D.1 The Brownian Motion

Let $\mathbf{R} = (R_1, \cdots, R_d)$ be a random vector with $\langle \mathbf{R} \rangle = \mathbf{0}$ and $\langle R^i R^j \rangle = a^2 \Lambda^{ij}$. Now define

$$\mathbf{X}(N) \equiv \sum_{n=1}^{N} \mathbf{R}_n$$

where $\{\mathbf{R}_n\}$ is an i.i.d. sequence. We have

$$\tilde{\rho}_N(\mathbf{k}) = (2\pi)^{-d} \langle \exp(-i\mathbf{k}.\mathbf{X}(N)) \rangle = (2\pi)^{-d/2} \langle \exp(-i\mathbf{k}.\mathbf{R}) \rangle^N = (2\pi)^{-d/2} \left(1 - \frac{a^2}{2} \Lambda^{ij} k_i k_j + \cdots \right)^N$$

In the large N limit

$$\tilde{\rho}_{N \gg 1}(\mathbf{k}) \approx (2\pi)^{-d/2} \exp\left(-\frac{Na^2}{2} \Lambda^{ij} k_i k_j\right)$$

Now as an aside, let's consider the diffusion problem

$$\partial_t \psi = \frac{1}{2} \Lambda^{ij} \partial_i \partial_j \psi$$

The Fourier impulse response is

$$\tilde{\psi}(\mathbf{k}) = (2\pi)^{-d/2} \exp\left(-\frac{t}{2} \Lambda^{ij} k_i k_j\right)$$

We see that if we let $N \to \infty$ and $a \to 0$ such that $t = Na^2$ remains constant, then the density function for the random walker mimics a solution of the diffusion differential equation. The resulting scaled path is the solution to the *stochastic* differential equation

$$d\mathbf{X}_t = \boldsymbol{\eta}(t) dt$$

with $\boldsymbol{\eta}(t)$ being the white Gaussian noise with the correlation function

$$\langle \eta^i(t) \eta^j(t') \rangle = \Lambda^{ij} \delta(t - t')$$

This path (at least when Λ^{ij} is the identity matrix) is called the Brownian motion.

Let us now answer some questions about the Brownian motion. For example, how much time, on average, does it spend in a region $\mathcal{V}$? That is

$$\langle T(\mathcal{V}) \rangle = \int_0^\infty dt \, \langle \mathbb{I}_{\mathcal{V}}(\mathbf{X}_t) \rangle = \int_{\mathcal{V}} d\mathbf{x} \int_0^\infty dt \, \rho_{\mathbf{X}_t}(\mathbf{x})$$

Which clearly suggest we take a closer look at

$$G(\mathbf{x}) \equiv \int_0^\infty dt\, \rho_{\mathbf{X}_t}(\mathbf{x})$$

This is the expected time that the path spends in a $d\mathbf{x}$ vicinity of the point $\mathbf{x}$ divided by the volume element $d\mathbf{x}$. Interestingly, this satisfies

$$\frac{1}{2}\Lambda^{ij}\partial_i\partial_j G(\mathbf{x}) = \int_0^\infty dt\, \partial_t \rho_{\mathbf{X}_t}(\mathbf{x}) = -\delta(\mathbf{x})$$

which is why we called it $'G'$ in the first place. This already means that for $d \leq 2$ the function $G(\mathbf{x})$ is infinite everywhere since there are no positive Green functions to the negative Laplacian operator in those dimensions. We can also see this explicitly by writing

$$G(\mathbf{x}) = \left[(2\pi)^d \, \det\left(\Lambda^{ij}\right)\right]^{-1/2} \left(\frac{1}{2}\Lambda_{ij}x^i x^j\right)^{1-d/2} \int_0^\infty du\, u^{d/2-2}\, e^{-u}$$

where Λ_{ij} is the inverse of Λ^{ij}. For $d \leq 2$, this is infinite, otherwise, the integral is just $\Gamma(d/2-1)$.

Bibliography

[1] Michael E. Peskin and Daniel V. Schroeder, *An Introduction to Quantum Field Theory* 41 (2019): 190088.

[2] Mark Srednicki...