The Rigid Sphere and The Ant

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If $\mathbf{c}(t)$ denotes the location of the center of the sphere and $\mathbf{x}(t)$ does so for the location of the ant, then

$$\mathbf{x}(t) = \mathbf{c}(t) + R(t)\mathbf{n}\Big(\theta(t), \varphi(t)\Big)$$

where $\mathbf{n}$ is the normal vector in the (θ, φ) direction. The following vector is conserved

$$\mathbf{c} + \alpha\big[\mathbf{c} + R\mathbf{n}\big]$$

and therefore

$$\mathbf{c} = \frac{\alpha}{1+\alpha}\big(\mathbf{n}_0 - R\mathbf{n}\big)$$

this yields the ant's position as

$$\mathbf{x} = \frac{\alpha\mathbf{n}_0 + R\mathbf{n}}{1+\alpha}$$

Then, we can write the angular momentum as

$$\mathbf{L} = \mathbf{L}_{\text{Sphere}} + \mathbf{L}_{\text{Ant}} = \beta\boldsymbol{\omega} + \mathbf{c}\times\dot{\mathbf{c}} + \alpha\mathbf{x}\times\dot{\mathbf{x}}$$

$$= \beta\boldsymbol{\omega} + \frac{\alpha}{1+\alpha}\big[\boldsymbol{\omega} + R(\mathbf{n}\times\dot{\mathbf{n}}) - (\boldsymbol{\omega}.R\mathbf{n})R\mathbf{n}\big]$$

$$= \beta\hat{\mathbf{z}} + \frac{\alpha}{1+\alpha}\big[\hat{\mathbf{z}} - \mathbf{n}_0\cos\theta_0\big]$$

where in the last line, I have used the initial conditions to evaluate the angular momentum. This is a linear system of equations to determine $\boldsymbol{\omega}$. Then, the evolution for R is found as

$$\frac{dR}{dt} = \Omega R = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} R$$

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It is best to use the Lagrangian method. Using

$$|\boldsymbol{\omega}|^2 = \frac{1}{2}\operatorname{Tr}\big(\Omega^T\Omega\big) = \frac{1}{2}\operatorname{Tr}\Big(\dot{R}^T\dot{R}\Big)$$

the Lagrangian is

$$L = \frac{1}{2}\Big[\frac{\beta}{2}\operatorname{Tr}\Big(\dot{R}^T\dot{R}\Big) + \frac{\alpha}{1+\alpha}|\Omega R\mathbf{n} + R\dot{\mathbf{n}}|^2\Big]$$

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The zeroth order solution is

$$\boldsymbol{\omega}_0(t) = \hat{\mathbf{z}} \quad ; \quad \Omega_0(t) = \begin{pmatrix} & -1 & \\ +1 & & \\ & & \end{pmatrix}$$

$$R_0(t) = \begin{pmatrix} \cos t & -\sin t & 0 \\ +\sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

therefore the first order solution will look as below

$$\boldsymbol{\omega}(t) = \hat{\mathbf{z}} + \alpha \boldsymbol{\omega}_1$$

$$R(t) \approx (1 + \alpha \Lambda) R_0(t)$$

where

$$\Lambda = \begin{pmatrix} 0 & -\lambda_3 & \lambda_2 \\ \lambda_3 & 0 & -\lambda_1 \\ -\lambda_2 & \lambda_1 & 0 \end{pmatrix}$$

In fact the vector $\boldsymbol{\lambda}$ describes the small extra rotation on the sphere due to the presence of the ant. The evolution of the rotation matrix $\dot{R} = \Omega R$ yields

$$\dot{\Lambda} = [\Omega_0, \Lambda] + \Omega_1$$

or in vector notation

$$\dot{\boldsymbol{\lambda}} = \hat{\mathbf{z}} \times \boldsymbol{\lambda} + \boldsymbol{\omega}_1$$

which is solved as

$$\boldsymbol{\lambda}(t) = \int_0^t ds\; R_0(t-s) \boldsymbol{\omega}_1(s)$$

Finally, the conservation of angular momentum leads to

$$\boldsymbol{\omega}_1 = \beta^{-1} \big[\cos\theta\, R_0 \mathbf{n} - \mathbf{n}_0 \cos\theta_0 - R_0 (\mathbf{n} \times \dot{\mathbf{n}}) \big]$$

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To find the net effect of all moving vehicles, let us first find the net correction to the angular velocity vector

$$\boldsymbol{\omega}_1 = \beta^{-1} \Big[R_0 \sum_i \cos\theta_i \mathbf{n}_i - \sum_i \mathbf{n}_{0,i} \cos\theta_{0,i} \; - R_0 \sum_i \mathbf{n}_i \times \dot{\mathbf{n}}_i \Big]$$

It is then natural to assume that the average vector

$$\sum_i \cos\theta_i \mathbf{n}_i = N \boldsymbol{\mu}$$

is time invariant. Here N is the number of all vehicles on the road. The z component of this vector is irrelevant and gets cancelled between the first two terms. The x and y components can in principle contribute to a precession motion of the earth; we neglect that since it is balanced out by other constant material like mountains. This means we are left with the final term. Regrading this, I assume a stationary motion on the roads and replace the integral in $\boldsymbol{\lambda}$ with a t. Finally, the net rotation is

$$\boldsymbol{\psi}(t) = \frac{-t}{\beta} R_0(t) \sum_i \alpha_i \mathbf{n}_i \times \dot{\mathbf{n}}_i$$

For each car moving up a road, let's say there is an adjacent car that moves down the road with the same speed. If the width of the road is w, (count negative when cars drive left like in the GB!) then

$$\boldsymbol{\psi}(t) = -\frac{Nt}{2\beta} R_0(t) \Big\langle wv\alpha\mathbf{n} \Big\rangle$$

The z-component leads to a change in the daytime

$$\delta D = \frac{ND^2}{4\pi\beta M a^2} \Big\langle wvm\cos\theta \Big\rangle$$

where D is the day time. To get some numbers, I assume

$$\langle wmv\cos\theta\rangle = 2\langle w\rangle\langle m\rangle\langle v\rangle\langle\cos\theta\rangle$$

$$\langle w\rangle \approx 10m \ ; \ \langle m\rangle \approx 1500\,kg \ ; \ \langle v\rangle \approx 15m/s \ ; \ \langle\cos\theta\rangle \approx 0.5 \ ; \ N \approx 10^9 \ ; \ \beta \approx 0.3$$

leading to

$$\boxed{\delta D \approx +2\times 10^{-15}\,s}$$

Assuming a rotation rate of smaller size in the x-y directions, (due to imbalances from the distribution of roads and their regulations) it turns out that it will take about 10^{19} days before the earth has rotated a hundredth of a radian in a direction other than its axis. Moral of the story: You are safe to ignore the effects of left/right driving conventions on the earth's motion!