CFT, The Yellow Book
A Solution Manual

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Chapter 2: Quantum Field Theory

Exercise 2.1

Since the operator $\mathcal{D}$ can involve higher order derivatives, the eigenfuncions usually need degeneracy indices as well.

$$\int dx\, u^*_{n,s} u_{m,r} = \delta_{mn}\delta_{rs}$$

First define

$$\tilde{\phi}_{n,s} \equiv \int dx\, u^*_{n,s}(x)\phi(x)$$

$$\tilde{\pi}_{n,s} \equiv \int dx\, u^*_{n,s}(x)\pi(x)$$

These operators are such that $\tilde{\phi}_{n,s}$ and $\tilde{\pi}^\dagger_{n,s}$ are conjugate. Furthermore the Hamiltonian becomes

$$H = \frac{1}{2}\int dx\, \left(\pi^2 - \phi\mathcal{D}\phi\right)$$

$$= \frac{1}{2}\sum_{n,s}\left(\tilde{\pi}^\dagger_{n,s}\tilde{\pi}_{n,s} + \omega_n^2\tilde{\phi}^\dagger_{n,s}\tilde{\phi}_{n,s}\right).$$

Finally define

$$a_{n,s} \equiv \frac{\omega_n\tilde{\phi}_{n,s} + i\tilde{\pi}_{n,s}}{\sqrt{2\omega_n}}$$

these operators satisfy

$$\boxed{[a_{n,s}, a^\dagger_{m,r}] = \delta_{mn}\delta_{rs}}$$

Also, note that if $u_{n,s}$ form a complete basis, then so do $u^*_{n,s}$; this implies

$$\boxed{\phi(x) = \sum_{n,s}\frac{1}{\sqrt{2\omega_n}}\left(u_{n,s}(x)a_{n,s} + u^*_{n,s}(x)a^\dagger_{n,s}\right)}$$

finally, we write

$$\omega_n a^\dagger_{n,s} a_{n,s} = H_{n,s} - \frac{i}{2}\omega_n\big(\tilde{\pi}^\dagger_{n,s}\tilde{\phi}_{n,s} - \tilde{\phi}^\dagger_{n,s}\tilde{\pi}_{n,s}\big) = H_{n,s} - \frac{1}{2}\omega_n$$

therefore $H = \sum_{n,s} H_{n,s}$ becomes

$$\boxed{H = \sum_{n,s}\omega_n\big(a^\dagger_{n,s}a_{n,s} + \frac{1}{2}\big)}$$

Exercise 2.2

Under a change of variables $\phi(x) \to \phi'(x) = \phi(x) + \varepsilon\delta\phi(x)$, the value of an integral remains unchanged

$$\begin{aligned}\langle X\rangle = \frac{1}{Z}\int [d\phi]\, X e^{iS[\phi]} &= \frac{1}{Z}\int [d\phi'] X' e^{iS'[\phi']} \\ &= \frac{1}{Z}\int [d\phi]\Big|\frac{\partial\phi}{\partial\phi'}\Big|(X+\delta X)e^{iS+i\delta S}\end{aligned}$$

Whenever $\delta\phi(x)$ does *not* depend on the filed $\phi(x)$, the Jacobian becomes unity and, up to the first order (in ε), we get

$$\boxed{\langle\delta X\rangle + i\langle X\delta S\rangle = 0}$$

For $X = \phi(y)$, and $\delta\phi(x) = \delta(x-z)$, we have $\delta X = \varepsilon\delta(y-z)$ and

$$\delta S = -\int dx\,\Big(m^2\phi(x)\delta(x-z) + \phi^{,\mu}(x)\partial_\mu\delta(x-z)\Big) = (\Box - m^2)\phi(z)$$

which yields[1]

$$\boxed{(\Box_z - m^2)\langle\phi(y)\phi(z)\rangle = i\delta(y-z)}$$

Exercise 2.3

REMARK: This is only valid for even potentials.

Essentially, we need to show

$$\{\psi_i,\psi_j\}\frac{\partial V}{\partial\psi_j} \overset{!}{=} [\psi_i, V(\psi)]$$

For the trivial even term, i.e. a constant potential, this obviously holds. For a generic even term and using the fact that the anti-commutator is a c-number, we may write

$$\begin{aligned}&\{\psi_i,\psi_j\}\partial_j\psi_{k_1}\psi_{l_1}\cdots\psi_{k_n}\psi_{l_n} \\ &= \sum_{r=1}^{n}\delta_{jk_r}\psi_{k_1}\psi_{l_1}\cdots\psi_{k_{r-1}}\psi_{l_{r-1}}\{\psi_i,\psi_j\}\psi_{l_r}\psi_{k_{r+1}}\psi_{l_{r+1}}\cdots\psi_{k_n}\psi_{l_n} \\ &\quad - \sum_{r=1}^{n}\delta_{jl_r}\psi_{k_1}\psi_{l_1}\cdots\psi_{k_{r-1}}\psi_{l_{r-1}}\psi_{k_r}\{\psi_i,\psi_j\}\psi_{k_{r+1}}\psi_{l_{r+1}}\cdots\psi_{k_n}\psi_{l_n} \\ &= \sum_{r=1}^{n}\psi_{k_1}\psi_{l_1}\cdots\psi_{k_{r-1}}\psi_{l_{r-1}}\big(\psi_i\psi_{k_r}\psi_{l_r} + \psi_{k_r}\psi_i\psi_{l_r} - \psi_{k_r}\psi_i\psi_{l_r} - \psi_{k_r}\psi_{l_r}\psi_i\big)\psi_{k_{r+1}}\psi_{l_{r+1}}\cdots\psi_{k_n}\psi_{l_n} \\ &= [\psi_i, \psi_{k_1}\psi_{l_1}\cdots\psi_{k_n}\psi_{l_n}] \;\blacksquare\end{aligned}$$

[1] Note that I am using a $-+++$ signature.

Exercise 2.4

To be added!

Exercise 2.5

The relevant term is

$$\langle \theta_{i_1}\theta_{i_2}\theta_{i_3}\theta_{i_4}\rangle = \frac{1}{8} A^{-1}_{j_1 j_2} A^{-1}_{j_3 j_4} \partial_{i_4}\partial_{i_3}\partial_{i_2}\partial_{i_1} b_{j_1} b_{j_2} b_{j_3} b_{j_4}$$

$$= \frac{1}{8}\langle \theta_{j_1}\theta_{j_2}\rangle\langle \theta_{j_3}\theta_{j_4}\rangle \sum_{p\in S_4} (-)^{\sigma(p)} \delta_{j_1 i_{p(1)}}\delta_{j_2 i_{p(2)}}\delta_{j_3 i_{p(3)}}\delta_{j_4 i_{p(4)}}$$

where $\sigma(p)$ is the number of adjacent permutations in the permutation p. Each Wick grouping gets repeated 8 times, one factor of two for internal order of each grouping and an extra factor of two for the order of the pairs. Symmtery of the ordinary multiplication and the antisymmetry of Grassmann multiplication along with antisymmetry of A^{-1}_{ij}, makes all these terms have the same sign and therefore

$$\langle \theta_{i_1}\theta_{i_2}\theta_{i_3}\theta_{i_4}\rangle = \pm\langle \theta_{i_1}\theta_{i_2}\rangle\langle \theta_{i_3}\theta_{i_4}\rangle \pm \langle \theta_{i_1}\theta_{i_3}\rangle\langle \theta_{i_2}\theta_{i_4}\rangle \pm \langle \theta_{i_1}\theta_{i_4}\rangle\langle \theta_{i_2}\theta_{i_3}\rangle$$

counting the adjacent permutations, the signs turn out to be

$$\boxed{\langle \theta_{i_1}\theta_{i_2}\theta_{i_3}\theta_{i_4}\rangle = +\langle \theta_{i_1}\theta_{i_2}\rangle\langle \theta_{i_3}\theta_{i_4}\rangle - \langle \theta_{i_1}\theta_{i_3}\rangle\langle \theta_{i_2}\theta_{i_4}\rangle + \langle \theta_{i_1}\theta_{i_4}\rangle\langle \theta_{i_2}\theta_{i_3}\rangle}$$

Exercise 2.6

Before starting to solve the problem, let us mention that the order of appearance of variables in $d\bar\theta$ is the opposite of the order in $d\theta$; so

$$\int d\bar\theta d\theta \equiv \frac{\partial}{\partial\bar\theta_1}\cdots\frac{\partial}{\partial\bar\theta_n}\frac{\partial}{\partial\theta_n}\cdots\frac{\partial}{\partial\theta_1}.$$

Now, back to the problem at hand, let us start by writing

$$e^{-M_{ij}\bar\theta_i\theta_j} = \sum_{n=0}^{\infty}\frac{(-)^n}{n!}\sum_{\substack{i_1\cdots i_n\\ j_1\cdots j_n}}\left(M_{i_1 j_1}\bar\theta_{i_1}\theta_{j_1}\right)\cdots\left(M_{i_n j_n}\bar\theta_{i_n}\theta_{j_n}\right)$$

One can permute the parantheses without changing the sign and this will cancel the $n!$ in the denominator.

$$e^{-M_{ij}\bar\theta_i\theta_j} = \sum_{n=0}^{\infty}(-)^n \sum_{\langle(i_1,j_1),\cdots,(i_n,j_n)\rangle}\left(M_{i_1 j_1}\bar\theta_{i_1}\theta_{j_1}\right)\cdots\left(M_{i_n j_n}\bar\theta_{i_n}\theta_{j_n}\right)$$

where the notation used in the second sum means that the sum is over unordered tuples of ordered pairs. Considering that no second powers exist in the realm of Grassmann variables, this is clearly the same as

$$e^{-M_{ij}\bar\theta_i\theta_j} = \prod_{(i,j)}\left(1 - M_{ij}\bar\theta_i\theta_j\right)$$

After taking the full integral (derivative) only the terms in which every variable is present exactly once, survives. That is

$$\int d\bar\theta d\theta\, e^{-M_{ij}\bar\theta_i\theta_j} = \sum_{p\in S_n} \pm M_{p(1)1}\cdots M_{p(n)n}$$

To commpute the signs, first (freely) order each term by their non-barred index to get

$$(-)^n M_{p(1)1}\cdots M_{p(n)n}\int d\bar\theta d\theta\, \bar\theta_{p(1)}\theta_1\cdots\bar\theta_{p(n)}\theta_n$$

$$= (-)^{\sigma(p)} M_{1p(1)} \cdots M_{np(n)}$$

Here is how the *minus-sign-counting* works: first, we have n minus signs from the product of all the $\left(1 - M_{ij}\bar{\theta}_i\theta_j\right)$ terms; then we have $1 + 2 + \cdots + n$ minus signs used to bring each θ_i to the far left before integration; and then at last, there are $\sigma(p) + 0 + 1 + \cdots + (n-1)$ minus signs to order the barred variables properly. Finally we recognize this as

$$\boxed{I_2(M) = \mathrm{det} M}$$

Chapter 3: Statistical Mechanics

Exercise 3.1

a)

$$\langle N_L \rangle = \sum_n \frac{N!}{n!(N-n)!} 2^{-N}.n = 2^{-N} N \sum_{m:=n-1} \frac{(N-1)!}{m!(N-1-m)!}$$
$$= 2^{-N} N.2^{N-1} = \boxed{N/2}$$

b)

$$\langle N_L^2 \rangle = \langle N_L \rangle + \langle N_L(N_L-1) \rangle$$
$$= \frac{N}{2} + \sum_n \frac{N!}{n!(N-n)!} 2^{-N}.n(n-1)$$
$$= \frac{N}{2} + 2^{-N} N(N-1) \sum_{m=n-2} \frac{(N-2)!}{m!(N-2-m)!}$$
$$= \frac{N}{2} + \frac{N(N-1)}{4} = \frac{N(N+1)}{4}$$

therefore

$$\boxed{\Delta N_L = \sqrt{\langle N_L^2 \rangle - \langle N_L \rangle^2} = \frac{\sqrt{N}}{2}}$$

c) Defining

$$X \equiv \frac{N_L - N/2}{\sqrt{N}/2}$$

and in the large N limit, we may write

$$f_X(x) \frac{2}{\sqrt{N}} \approx \mathbb{P}\Big[N_L = n(x)\Big]$$

where

$$n(x) \equiv \lfloor \frac{N + x\sqrt{N}}{2} \rfloor \approx \frac{N + x\sqrt{N}}{2}$$

Using the Stirling formula (note that $\lim_{N\to\infty} n(x) = \infty$) we get

$$\boxed{f_X(x) = \frac{1}{\sqrt{2\pi} e^{x^2/2}}}$$

Exercise 3.2

a)

$$Z = \sum_{[\sigma]} \exp\Big\{ \sum_{n=0}^{N-1} \big[-(H/2)(\sigma_n + \sigma_{n+1}) + K\sigma_n\sigma_{n+1} \big] \Big\}$$
$$= \sum_{[\sigma]} T_{\sigma_n,\sigma_{n+1}} = \mathrm{Tr}\, T^N$$

with

$$T_{\sigma,\sigma'} \equiv \exp\Big\{ \sum_{n=0}^{N-1} \big[-(H/2)(\sigma + \sigma') + K\sigma\sigma' \big] \Big\}$$

or, in matrix form

$$\boxed{T \equiv \begin{pmatrix} e^{K-H} & e^{-K} \\ e^{-K} & e^{K+H} \end{pmatrix}}$$

b)

$$\begin{aligned} f &\equiv \lim_{N\to\infty} -\frac{1}{N} \log \mathrm{Tr}\, T^N \\ &= -\log\Big[\lim_{N\to\infty} (\tau_+^N + \tau_-^N)^{1/N}\Big] \\ &= -\log \tau_+ \end{aligned}$$

where

$$\tau_+ \equiv e^K \cosh H + \sqrt{e^{2K}\sinh^2 H + e^{-2K}}$$

therefore

$$\boxed{f = -\log\Big[e^K \cosh H + \sqrt{e^{2K}\sinh^2 H + e^{-2K}}\Big]}$$

c)

$$\boxed{M \equiv \frac{\partial f}{\partial H} = -\frac{e^K \sinh H + e^{2K}\sinh(2H)/2\sqrt{e^{2K}\sinh^2 H + e^{-2K}}}{e^K \cosh H + \sqrt{e^{2K}\sinh^2 H + e^{-2K}}}}$$

for $H \ll 1$

$$\boxed{M \approx He^{2K}}$$

and therefore the magnetic susceptibility diverges like e^{2K} as temperature approaches zero. We also note that at finite temperatures, the free energy (per spin site) is well behaved and therefore we observe no phase transitions.

d) Using the symmetries of the problem, it is easy to show that the two point correlation function

$$C_\ell \equiv \langle \sigma_n \sigma_{n+\ell} \rangle - \langle \sigma^2 \rangle$$

is, up to a constant function of ℓ, the same as

$$2\Big[P_{++}^\ell(H,K) + P_{++}^\ell(-H,K)\Big]$$

where $P_{++}^\ell(H,K)$ is the probability that two spins, a distance ℓ apart, are both pointing up. This is written as

$$\begin{aligned} P_{++}^\ell(H,K) &= \lim_{N\to\infty} \frac{\langle +|\, T^\ell \,|+\rangle \langle +|\, T^N \,|+\rangle}{\mathrm{Tr}\, T^{N+\ell}} \\ &= \big|\, \langle \tau_+|+\rangle \langle +|\tau_-\rangle \,\big|^2 \Big(\frac{\tau_-}{\tau_+}\Big)^\ell + cte. \end{aligned}$$

I won't substitute the complete formulae; only note that since the spectrum of T does not depend on the sign of H, we know that the correlation function is always in the form

$$\boxed{C_\ell = C_0 \Big(\frac{1-\chi}{1+\chi}\Big)^\ell}$$

where

$$\chi \equiv \sqrt{\tanh^2 H + e^{-4K}\big(1 - \tanh^2 H\big)}$$

Exercise 3.3

a) Similar to what we did in the previous problem, the transfer matrix is

$$T_{ij} = e^{-\beta E_{ij}} = e^K \delta_{ij} + e^0 (1 - \delta_{ij})$$

$$\boxed{T = (e^K - 1)\mathbb{I} + J}$$

b) J has two different eigenvalues, one is q (non-degenerate) and the other is zero ($q-1$ fold degenerate). Adding a multiple of identity, we find that the largest eigenvalue is

$$\lambda_{max} = e^K + q - 1$$

and

$$\boxed{f = -\log\left(e^K + q - 1\right)}$$

Exercise 3.4

The transfer matrix has two indices, each of which are strings of signs $\pm$ of length L.

$$\boxed{T_{\mathbf{s},\mathbf{s}'} = \exp\left\{K\left[\mathbf{s}.\mathbf{s}' + \frac{1}{2}\sum_{i=0}^{L-1}\left(s_i s_{i+1} + s'_i s'_{i+1}\right)\right]\right\}}$$

Exercise 3.5

a) This is a weak version of the so called Perron-Frobenius theorem. For a real symmetric matrix with positive entries, let the diagonalisation take the form $A = Q\Lambda Q^T$. Now consider the following optimization problem

$$\max_{||\mathbf{x}||_2=1} ||A\mathbf{x}||_2^2 = \max_{||\mathbf{x}||_2=1} \mathbf{x}^T Q\Lambda Q^T Q\Lambda Q^T \mathbf{x}$$

$$= \max_{\substack{||\mathbf{y}||_2=1 \\ \mathbf{y}\equiv Q^T\mathbf{x}}} ||\Lambda\mathbf{y}||_2^2 = \lambda_{\max}^2$$

The first thing to prove is that the maximum is *never achieved* for vectors having components of both positive and negative sign; this is seen using the strict inequality

$$||A\mathbf{x}||_2^2 = \sum_i \left(\sum_j A_{ij} x_j\right)^2$$

$$< \sum_i \left(\sum_j |A_{ij} x_j|\right)^2 = \sum_i \left(\sum_j A_{ij} |x_j|\right)^2$$

This also implies that the largest singular value is unique, otherwise, we would have $||A\mathbf{x}|| = \lambda_{max}$ on a great circle around the unit sphere and this - inevitably - includes vectors with both positive and negative components. Therefore, the largest eigenvalue of a symmetric, (entry-wise) positive matrix is 1. positive; 2. unique; 3. corresponding to a positive eigenvector.

b) In the eigenvector basis, this map is as follows

$$y_i \to \frac{\lambda_i y_i}{\sqrt{\sum_j \lambda_j^2 y_j^2}}$$

After N iterations, this is parallel to

$$\left(y_1^0,\, (\frac{\lambda_2}{\lambda_1})^N y_2^0,\, (\frac{\lambda_3}{\lambda_1})^N y_3^0,\, \cdots\right)^T$$

Which is clearly approaching the maximal eigenvector.

c) Here is the pseudo-code

```
1. Pick a random vector 'x' and normalize it.
2. Let L = |T*x|
3. While | (T*x/|T*x|) - x| > some threshold:
4.    x --> T*x/|T*x|
5.    L --> |T*x|
6. Report L and x.
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d) It is relatively easy to show that if the threshold in the algorithm above is called ε, then the uncertainty in estimating the logarithm of the largest eigenvalue is given by

$$\delta \log ||T||_2 \lesssim \frac{1+r}{1-r}\varepsilon^2$$

where r is the ratio

$$r(A) \equiv \frac{\lambda_2(T)}{\lambda_1(T)}$$

Using, $\varepsilon = 10^{-3}$ we can easily neglect this error in comparison with the systematic error coming from the $\mathcal{O}(L^{-2})$ terms; therefore it is best to use only the two largest values of L, namely $L = 11$ and $L = 12$ to get

$$\boxed{\hat{f}_0 = -0.929685\cdots \quad \text{and} \quad \hat{c} = 0.504\cdots}$$