Game Theory
(Solutions to the Problems)

April 17, 2023

Pset #1

Problem 2.5

Consider a set of outcomes $O = \{A, B, C\}$.

Simplification of lotteries

For a simple lottery, the utility is given by the expectation value

$$u([p_A, p_B, p_C]) = p_A - p_C$$

For a compound lottery, this is

$$u\big([q_i(L_i)]\big) = 2020 + \langle u \rangle$$

It should be clear that this satisfies all the other axioms.

From now on, the simplification axiom is assumed to hold. This means the utility function is defined on the probability simplex. *The other axioms are not formulated in their usual form. This made the other examples look bizarre!*

Continuity

Let the probabilities p_i be written as[1]

$$p_i = \sum_{n=1}^{\infty} B_{i,n} 2^{-n}, \quad B_{i,n} \in \{0, 1\}$$

Now define u to be

$$u(p_i) = \sum_{n=1}^{\infty} (4B_{1,n} + 2B_{2,n} + B_{3,n}) 8^{-n}$$
$$= .B_{1,1}B_{2,1}B_{3,1}B_{1,2}B_{2,2}B_{3,2} \cdots$$

This is clearly not continuous. (In fact this function is one to one and onto from $[0,1]^3$ to $[0,1]$.) The axiom of independence is irrelevant since no 2 lotteries are the same. To show the monotonicity property (on edges) take two different outcomes. The utility takes the form

$$a \sum_{n=1}^{\infty} B_n 8^{-n} + b \sum_{n=1}^{\infty} \bar{B}_n 8^{-n}$$
$$= C + D \sum_{n=1}^{\infty} B_n 8^{-n}$$

which is clearly monotone.

Another counter example is found by acting with a function of the form

$$f_n(x) = \frac{2}{n} x + \frac{n-2}{n} \theta(x - 1/2)$$

on a linear utility scaled to fit in the interval [0, 1]. (Figure it out!)//

[1] To make the representations unique we assume that for any number other than unity, infinitely many 0's exist in the expansion.

$$1 = .1111111 \cdots, \quad \frac{1}{2} = .1 \neq .01111111 \cdots$$

Monotonicity

Let $u(A) = 0$, $u(B) = 1$, $u = \frac{1}{2}$ otherwise. This is Continuous (any nontrivial lottery between A and B has the same utility as C) and (trivially) satisfies independence.

Independence

Let $u(A) = u(B) = u(C) = 1$ and $u = \sum_i p_i^2$ in between(simplification holds). Continuity is evident, and monotonicity is vacuously true. But in any direction on the probability simplex, there are equivalent lotteries. Substituting them with each other in a mixed lottery, leads to a movement of the equivalent simple lottery in that specific direction. This, in general will lead to a change in the utility function.

Problem 2.12

For $\alpha = 0$ this holds trivially. For $\alpha > 0$

$$\begin{aligned}
[\alpha(L_1), (1-\alpha)(L_3)] &\geq [\alpha(L_2), (1-\alpha)(L_3)] \\
\Rightarrow \alpha u(L_1) + (1-\alpha)u(L_3) &\geq \alpha u(L_2) + (1-\alpha)u(L_3) \\
\Rightarrow u(L_1) &\geq u(L_2) \\
\Rightarrow \alpha u(L_1) + (1-\alpha)u(L_4) &\geq \alpha u(L_2) + (1-\alpha)u(L_4) \\
\Rightarrow [\alpha(L_1), (1-\alpha)(L_4)] &\geq [\alpha(L_2), (1-\alpha)(L_4)]
\end{aligned}$$

Problem 2.17

Yes; in fact there are many. One example is

$$u = \langle 1001x + y \rangle$$

Problem 2.21

Since the preference relation induced by both functions u and v is the same, $u(\vec{p}) = u(\vec{q})$ implies $v(\vec{p}) = v(\vec{q})$. In other words v is a function of u. Linearity (implied by VN-M axioms) implies an affine (where does the constant term come from?) form

$$v = au + b$$

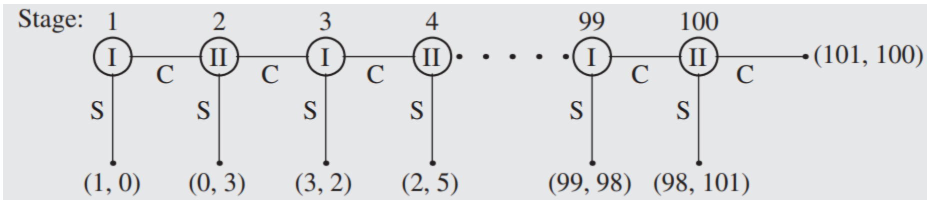
Problem 2.25

For a risk averse player, the utility function is concave (in terms of the label X). Using Jensen's inequality

$$\mathbb{E}\big[U(X)\big] = \mathbb{E}_X\Big[U(X + \mathbb{E}_Y Y)\Big] \geq \mathbb{E}_X \mathbb{E}_Y U(X + Y)$$

Pset #2

Problem 3.12



We use "Backward Induction" to solve the game:

Proposition1: if I(II) is not going to cooperate in the next round, II(I) is better off stopping the game now! (proof by inspection)

Proposition2: II will definitely stop the game in the last round. (Proof: Effectively, it is given that I stops the game in the round 101.)

The "rational" implication of the two propositions (using a few intermediary steps) is that I is better off stopping the game in the first round.

REMARK: We'll see how to fix this later on (Cf. repeated games).

Problem 3.14

a) Yes the game is of complete information and finite.

b) Start from the leftmost column and find the leftmost column with an odd number of 1's; say it will be the Nth column. (You could neglect the previous columns completely from now on.)

$$X_i = \cdots X_{i,3}X_{i,2}X_{i,1}. \quad B = \bigoplus_i X_i = B_{i,N} \cdots B_{i,3}B_{i,2}B_{i,1}.$$

Let $\hat{X}$ be the number of matches in one of the piles with a 1 in the N^{th} column.

$$\hat{X} = \cdots \hat{X}_{N+1}1\hat{X}_{n-1} \cdots \hat{X}_1.$$

Now pick some number Y of matches from this pile such that the remainder reads

$$\hat{X} - Y = \cdots \hat{X}_{N+1}0B_{n-1}B_{n-2} \cdots B_1. < \hat{X}$$

This clearly leads to a winning position.

c) In a winning position, $B = 0$. Taking any positive number of matches, Y, leads to $B' = B - Y \neq 0$

d) At the end of the game, the winner faces a situation in which $X_i = 0$. This is clearly a winning position.

e) Based on our previous reasonings it should be clear that player II has a winning strategy if and only if the game starts in a winning situation.

Problem 3.20

a)

$$G = (X_0 + X_1)(X_2 + X_3)$$

For both players it is possible to ensure that at least one of the variables in each set $\{X_0, X_1\}$ and $\{X_2, X_3\}$ has the value 1, therefore making the game value $G = 1$. Clearly this is what player I is interested in and therefore he has a winning strategy.

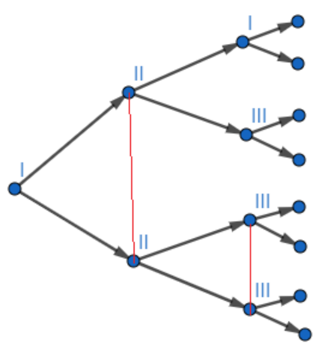
b) A $2k$ level game consists of $2^{2(k-1)}$ independent (parallel) games of depth 2.

$$G = \Big(\big((G_0 + G_1)(G_2 + G_3) + (G_4 + G_5)(G_6 + G_7)\big) \cdots + G_{2^{2(k-1)}})\big)\Big)$$

As we saw in the previous part, it is possible for both players to ensure that all G_i become 1 therefore making $G = 1$. This is only in I's interest and therefore he has a winning strategy.

Problem 3.24

a)



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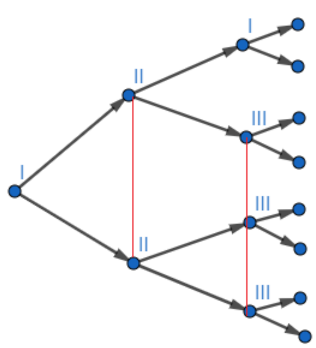

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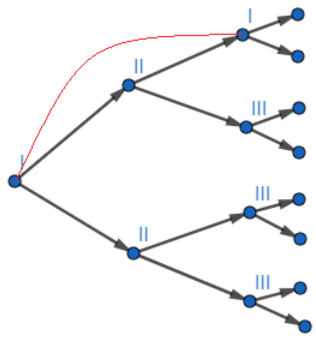
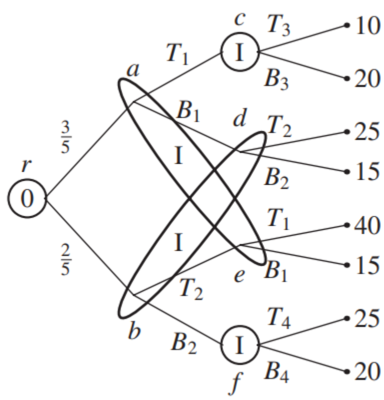


Problem 3.27



Here a proposition is called "known" iff there is no doubt about its occurrence; partial information in terms of probability distributions don't count!

a) $\{a, e\}$: I does not know if it is her first or second time making a decision. She also doesn't know the outcome of the coin. She knows that she is either at vertex a or at vertex e. Something similar may be said about $\{b, d\}$. For $\{c\}$ and $\{f\}$, she knows "everything" (her exact position in the game). (Whether she has made a choice before or not, the outcome of the coin, etc.)

b) There are 4 information sets with 2 possible actions at each one $\Rightarrow 2^4 = 16$ possible strategies. eg. $T_1B_2B_3B_4$.

c) The book says the numbers determine the payment and not the utility; I assume risk-neutrality to deduce the utility as well. B_3 and T_4 definitely seem to be rational choices in the corresponding information sets. To determine the rest, let us compute the expected utilities

$$u\big(T_1T_2B_3T_4\big) = \frac{3}{5} \times 20 + \frac{2}{5} \times 40 = 28$$

$$u\big(T_1B_2B_3T_4\big) = \frac{3}{5} \times 20 + \frac{2}{5} \times 25 = 22$$

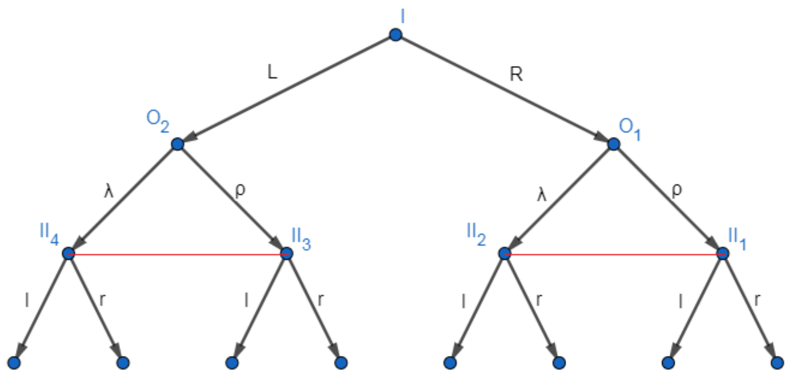
$$u\big(B_1T_2B_3T_4\big) = \frac{3}{5} \times 25 + \frac{2}{5} \times 15 = 21$$

$$u(B_1B_2B_3T_4) = \frac{3}{5} \times 15 + \frac{2}{5} \times 25 = 19$$

Which clearly suggest $T_1T_2B_3T_4$ to be the optimal strategy.

Problem 3.38

a)



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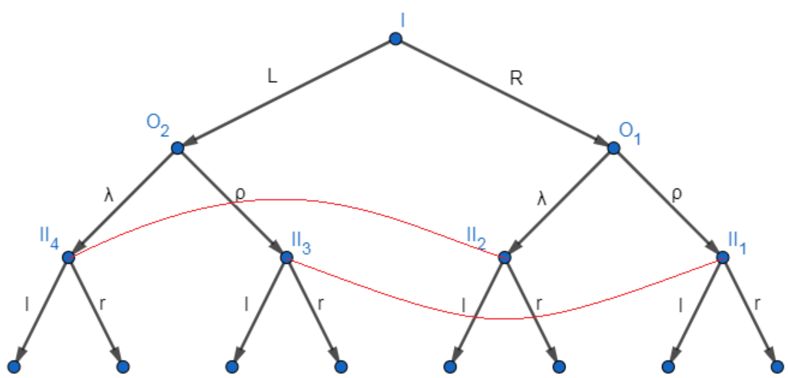

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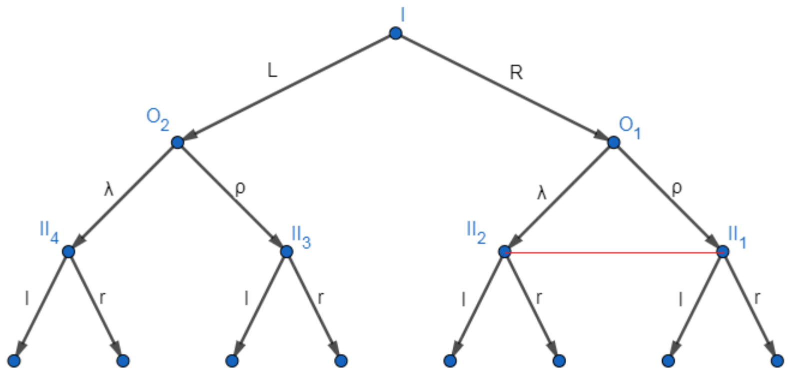
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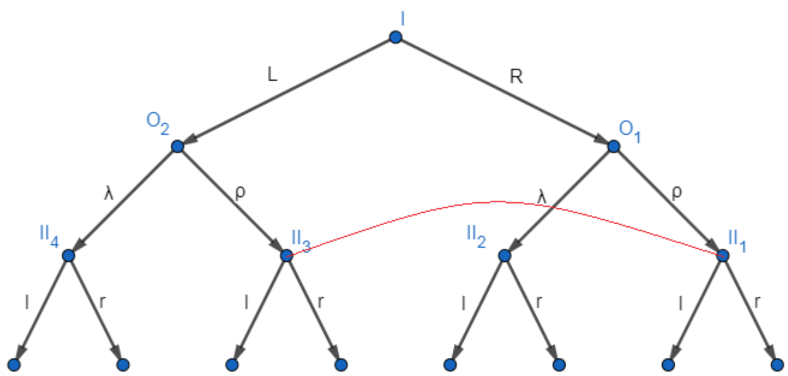
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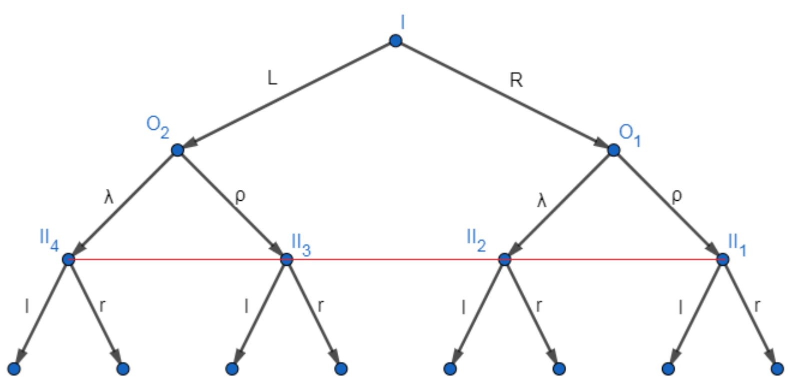
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Pset #3

Problem 4.9

a) Proof of contradiction: Let us assume this is not the case. This means that at some point, one of the eliminated strategies may have provided a beneficial (or at least not lossy) deviation for some player. (Induction is used implicitly in this reasoning.)

b) For a strict Nash equilibrium, $\vec{s}^*$, none of the strategies played with positive probability are dominated (by definition.) This means after one iteration, the strategy is not eliminated. But since eliminating a few deviation options does not change the "strict Nash equilibrium" condition, $\vec{s}^*$ is a strict equilibrium for the new reduced game. Induction may be used to prove the desired result.

Problem 4.17

Let us start by looking for dominated strategies. None exists! Ok then the general strategy would be to find best responses and find consistent solutions to the set of equations

$$s_i = b_i(s_{-i})$$

Let us start by player III's best response

$$b_{III}(Aa) = \alpha; \quad b_{III}(Ab) = ?$$

$$b_{III}(Ba) = ?; \quad b_{III}(Ab) = \gamma$$

Ha! β is not the best response to any thing. Unless $Ab\beta$ or $Ba\beta$ form equilibria; but they don't. (I has better options in both situations.) Hence we eliminate β.

Now A is weakly dominated. ($\boxed{Ab\gamma}$ and $\boxed{Ab\alpha}$ are equilibria under which A is played with positive probability.)

Eliminating A, yields $\boxed{Ba\alpha}$ and $\boxed{Ba\gamma}$ as the last equilibria.

Problem 4.28

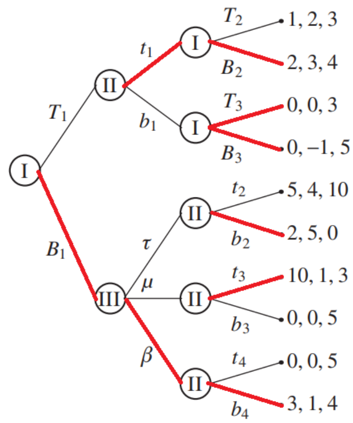
1, 0	1, 1
0, 0	0, 1
1, 0	2, 0

Eliminating the "fraud" strategy (Bottom) from I's set of actions, makes TR an equilibrium.

Problem 4.36

$$\text{Optimality for I} \Leftrightarrow u(s_I^*, s_{II}^*) \geq u(s_I', s_{II}^*) \;\; \forall s_I' \in S_I$$
$$\text{Optimality for II} \Leftrightarrow u(s_I^*, s_{II}^*) \leq u(s_I^*, s_{II}') \;\; \forall s_{II}' \in S_{II}$$

Problem 4.40



This clearly suggests

$$\Big[p\big(B_1B_2B_3, t_1b_2t_3b_{,}\beta\big), (1-p)\big(B_1B_2T_3, t_1b_2t_3b_{,}\beta\big)\Big]$$

Other equilibria are obtained by changing the strategies in irrelevant manners. For example:

$$\big(B_1T_2B_3, t_1b_2t_3b_{,}\beta\big)$$

Problem 4.43

In general, this is FALSE! A game on the unit square may be considered as a general strategic form game, and therefore many counter examples exist.

Assuming that the utility functions are bilinear, without loss of generality (using positive affine transformation and possible involutions: $X \to 1-X$ or $Y \to 1-Y$) we may write

$$U_I(X,Y) = (X-a')(Y-b)$$

$$U_{II}(X,Y) = (X-a)(Y-b')$$

There is a Nash equilibrium in the interior of the unit square if $0 < a, b < 1$. But even then, there is no way that this is the unique Nash equilibrium since $(1,1)$ is guaranteed to be one too.

We have neglected some exceptional cases here but they are hardly worth mentioning and the reader can figure them out for herself.

Problem 4.47

a) Each driver faces a choice: which route to take? And the outcome (time) depends on his choice as well as others' choice. The utilities are not given but it is assumed that they are monotonically decreasing functions of time.

b) Indifference : $52 + 1.1x_T = 52 + 1.1x_B$. This yields $x_T = x_B = 30\,\text{min}^{-1} \Rightarrow T = 85$ min's.

c) Out of the x_T cars taking the top route, a fraction a use the new road. Indifference at any fork ways, helps us find the unknowns.

$$51 + .1x_T(1-a) = 71 + .1x_T a - x_T(1-a)$$

$$118 - x_T(1.1-a) = 52 + x_T(1.1 - .1a)$$

This solves to give $x_T = 40\,\text{min}^{-1}$, $a = \frac{1}{2}$ which makes for a time $T = 94$ min's!

Problem 5.9

(Game B) Checking the pure strategies, we find no equilibrium. Therefore the equilibria are mixed. Let II's strategy be $(p(L), (1-p)(R))$.

$$u_a = -11p + 8; \quad u_b = 7p - 2$$

$$u_c = -14p - 4; \quad u_d = 23p - 8$$

Looking at the graph, only the intersection $u_a = u_d$ is a best response. $\boxed{p = \frac{8}{17}}$, $\boxed{v = \frac{48}{17}}$. It remains to calculate q $\Big(\big[q(a), (1-q)(d)\big]\Big)$ such that player II becomes indifferent. $\Rightarrow \boxed{q = \frac{11}{34}}$.

(Game D) We act similarly to find $\boxed{v = 5}$ as the value of the game under

$$\vec{s} = \Big(\big[\frac{2}{3}(T), \frac{1}{3}(B)\big], \big[x(L), (2-3x)(M), (2x-1)(R)\big]\Big); \quad \forall x \in \big[\frac{1}{2}, \frac{2}{3}\big]$$

Problem 5.13

a) This is trivial! $(u(x,y) = -u(y,x))$

b) Under the optimal strategy we have $\langle u \rangle \leq v = 0$. But this means

$$\mathbb{P}[X \neq Y \& (X,Y) \in E] \leq \frac{1}{2}$$

holds for any strategy of player I i.e. any distribution for X. Specifically the pure distributions give

$$\sum_{\{y \in V:\ (y,x) \in E\}} q(y) \geq \frac{1}{2}$$

Problem 5.15

In mathematical notation, the assumption reads

$$u(\sigma, \sigma_\backslash) > u(\hat{\sigma}, \sigma_\backslash) \quad \forall \sigma_\backslash \in \Delta_\backslash$$

where $\sigma_\setminus$ denotes other players' strategy.

using linearity of any of the functions $u(., \sigma_\setminus)$, the assumption is equivalent to

$$\partial_{\sigma-\hat{\sigma}} u(\sigma', \sigma_\setminus) > 0 \quad \forall(\sigma', \sigma_\setminus) \in \Delta \times \Delta_\setminus$$

a) This is false; the game

0, 0	0, 0
-2, 0	1, 0
1, 0	-2, 0

with

$$\hat{\sigma} = (0, \frac{1}{2}, \frac{1}{2})^T, \quad \sigma = \hat{\sigma} + \epsilon(2, -1, -1)^T$$

for some small ϵ provides a counter example.

b) This is true; assuming optimality for a strategy σ^* means

$$\sigma^* + \epsilon(\sigma - \hat{\sigma}) \notin \Delta$$

But this vector is manifestly normalized to 1; therefore the positivity condition needs to be violated. In other words

$$\exists i \ \ s.t. \ \ \sigma_i^* = 0, \ \ \hat{\sigma}_i > \sigma_i \geq 0$$

which is what we wanted to prove.

Problem 5.20

a) Let us parametrize this strategy as $\left[x_T(T), x_M(M), x_B(B)\right]$. We need to find a solution to

$$\begin{pmatrix} 1 & 1 & 1 \\ 6 & -4 & -3 \\ -3 & 2 & -1 \end{pmatrix} \begin{pmatrix} x_T \\ x_M \\ x_B \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

This yields the (valid) solution

$$x_T = \frac{2}{5}; \ \ x_M = \frac{3}{5}; \ \ x_B = 0$$

b) Similarly

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & -9 & -4 \\ 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} y_L \\ y_C \\ y_R \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

which yields

$$y_L = \frac{22}{25}; \ \ y_C = \frac{2}{25}; \ \ y_R = \frac{1}{25}$$

d) We show that the equations considered in problem 5.54 hold: The distribution exists and has positive weights, the optimality of support conditions are met vacuously and finally the indifference is guaranteed by the equations we just solved.

c) This is implied by (d).

e) The following game won't be in contradiction with (d) since in (d) we assumed that both players have such an equalizing strategy.

1	1
0	0

Problem 5.23

a)

$$q^j = \alpha p^j + (1-\alpha)\hat{p}^j \geq 0; \quad \sum_j q^j = \alpha \sum_j p^j + (1-\alpha)\sum_j \hat{p}^j = 1$$

b)

$$u(\tau_I, \sigma_{II}) = \sum_j q^j U(s_I^j, \sigma_{II}) = \alpha U(\sigma_I, \sigma_{II}) + (1-\alpha) U(\hat{\sigma}_I, \sigma_{II})$$

c)

$$U(\tau_I, \sigma_{II}) - v = \alpha\big(U(\sigma_I, \sigma_{II}) - v\big) + (1-\alpha)\big(U(\hat{\sigma}_I, \sigma_{II}) - v\big) \geq 0$$

d) This a special case of the previous part for v being the value of the game.

e) We just proved this (via showing that the definition holds.)

Problem 5.24

a)

$$p^{*,i} = \lim_{k\to\infty} p_k^i \geq 0; \quad \sum_i p^{*,i} = \sum_i \lim_{k\to\infty} p_k^i = \lim_{k\to\infty} \sum_i p_k^i = 1$$

b)

$$U(\sigma^*, .) - v = \lim_{k\to\infty} U(\sigma_k, .) - v \geq 0$$

c) This holds by definition.

Problem 5.26

Game A:

a) B and R are strictly dominant. Therefore the unique equilibrium is (B, R) with the utility $(1, 1)$. b) Once again dominance yields B and R as the answers. c) Playing a dominated strategy is never recommended, therefore B and R ARE recommended.

Game F:

a) There are 3 equilibria

$$(B, L) \to (4, 0)$$

$$(T, R) \to (3, 3)$$

$$\Big([\frac{2}{3}(T), \frac{1}{3}(B)], [\frac{1}{3}(L), \frac{2}{3}(R)]\Big) \to (\frac{8}{3}, \frac{4}{3})$$

b) For the first player $\left[\frac{1}{3}(T), \frac{2}{3}(B)\right]$ is the maxmin strategy and guarantees $\frac{8}{3}$ and for the second player $\left[\frac{2}{3}(L), \frac{1}{3}(R)\right]$ guarantees $\frac{4}{3}$

c) Any strategy that is a best response to some strategy may be considered a wise advise since there is no prior assumed.

Problem 5.27

a) This is a version of Rock-Paper-Scissors with all its symmetries. One may quickly verify that no pure equilibria exist. It is also possible that any distribution other than the uniform distribution over the actions has a pure best response. This proves the statement.

b) Because he has better options; namely R (assuming the problem is referring to C)

Problem 5.37

Let us denote the value of a ZSG corresponding to a utility matrix U_{ij} with $v(U)$. The goal is to prove

$$|v(U) - v(U')| \leq ||U - U'||_\infty$$

Clearly (why?) it suffices to show that this Lipschitz condition holds almost everywhere and that the function $v(U)$ is a continuous one. At every equilibrium of the game U, only a subset of the actions are played with positive probability. Almost everywhere, these form an invertible square submatrix of U which we will call A (why?). The reader should convince himself that the game value is continuous on the zero measure subset that the submatrix is not square. This means we are ready to deal with the main part of the proof; to show the Lipschitz condition for a square game with equalizing strategies on both sides. If we denote player I and II's strategies with $\vec{x}$ and $\vec{y}$ respectively we get

$$A\vec{y} = v\vec{1}; \quad \vec{x}^T A = v\vec{1}^T$$

Or

$$\boxed{v(A) = \frac{1}{\sum_{i,j} A^{-1}_{ij}}}$$

How does this quantity change if one changes A? Define

$$A(\lambda) = A_0 + \lambda A_1$$

All we need to show is

$$\Big| \lim_{\lambda \to 0} \frac{v(A(\lambda)) - v(A(0))}{\lambda \max_{i,j} |A_{1,ij}|} \Big| \leq 1$$

Taylor expanding the inverse $A^{-1}(\lambda) = A_0^{-1} + \lambda X + \mathcal{O}(\lambda^2)$ it is easy to show $X = -A_0^{-1} A_1 A_0^{-1}$. This in turn yields

$$\begin{aligned} v(A(\lambda)) &= \frac{1}{\sum_{ij} A_{ij}^{-1}(\lambda)} \\ &= v(A(0)) - \lambda \frac{\sum_{ij} X_{ij}}{\left(\sum_{ij} A_{0,ij}^{-1}\right)^2} + \mathcal{O}(\lambda^2) \end{aligned}$$

Which implies

$$\Big| \lim_{\lambda \to 0} \frac{v(A(\lambda)) - v(A(0))}{\lambda \max_{i,j} |A_{1,ij}|} \Big| = \Big| \frac{\sum_{ijkl} A_{0,ik}^{-1} A_{1,kl} A_{0,lj}^{-1}}{\max_{ij} |A_{1,ij}| \sum_{ijkl} A_{0,ik}^{-1} A_{0,lj}^{-1}} \Big| \leq 1$$

Problem 5.38

$$A_{ij} = i; \quad B_{ij} = m - i + 1$$

Problem 5.54

Equations 5.74 and 5.75 guarantee that the strategies are indeed feasible. Equations 5.76 to 5.79 *define* Y_i as the set of actions played with positive probability by player i. It remains to show that none of the players has better options. Equations 5.70 and 5.71 show that none of the players will benefit from changing the probabilities in his strategy while keeping the support fixed (indifference). And finally equations 5.72 and 5.73 guarantee that changing the support doesn't benefit any player.

Problem 5.60

a) Using the symmetry of the game, we clearly have

$$v \equiv \max_{\sigma_I \in \Sigma_I} \min_{\{\sigma_{II} \in \Sigma_{II};\ C(\sigma_I, \sigma_{II}) \leq \gamma\}} U(\sigma_I, \sigma_{II}) = - \min_{\sigma_{II} \in \Sigma_{II}} \max_{\{\sigma_I \in \Sigma_I;\ C(\sigma_I, \sigma_{II}) \leq \gamma\}} U(\sigma_I, \sigma_{II})$$

To compute the value v, let us note that the strategies B and R are strictly dominated. If I plays $\big[p(T), (1-p)(B)\big]$, then II will play $\big[q(L), (1-q)(R)\big]$ with maximum q allowed, namely $\frac{1}{2p}$. This yields a utility

$$u(p) = p - \frac{1}{2p}$$

Therefore I plays the strategy $p = 1$ which yields the value $v = \frac{1}{2}$.

b) There are only two equilibria and we just found them (Using the best response strategy this must be clear.)

c)

$$\max_{\sigma_I \in \Sigma_I} \min_{\{\sigma_{II} \in \Sigma_{II};\ C(\sigma_I, \sigma_{II}) \le \gamma\}} U(\sigma_I, \sigma_{II}) \ge \min_{\{\sigma_{II} \in \Sigma_{II};\ C(\sigma_I^*, \sigma_{II}) \le \gamma\}} U(\sigma_I^*, \sigma_{II}) \quad \forall \sigma_I^*$$

To prove the inequality, take σ_I^* to be the first argument in the optimization problem

$$\min_{\sigma_{II} \in \Sigma_{II}} \max_{\{\sigma_I \in \Sigma_I;\ C(\sigma_I, \sigma_{II}) \le \gamma\}} U(\sigma_I, \sigma_{II})$$

d)

$$\inf_{\{\sigma \in \Sigma:\ C(\sigma, \sigma_\setminus) \le \gamma\}} C(\sigma, \sigma_\setminus) \le \gamma \quad \forall \sigma \in \Sigma$$

Using the continuity (why is it continuous?) and the compactness of the feasible strategies, we get

$$\sup_{\sigma_\setminus \in \Sigma_\setminus} \inf_{\{\sigma \in \Sigma:\ C(\sigma, \sigma_\setminus) \le \gamma\}} C(\sigma, \sigma_\setminus) \le \gamma$$

e) For the sake of simplicity in notation, assume the following renamings

$$s_i \to X \sim \sigma; \quad s_{-i} \to Y \sim \mu$$

The constraint may be written as

$$\int c(x, y) d\sigma d\mu \le 0$$

Finally let us define

$$\Sigma_\mu \equiv \{\sigma | \int c(x, y) d\sigma d\mu \le 0\}$$

Now we are given $\mu_n \to \mu$ and σ such that $\sigma \times \mu$ satisfies the constraint.[2] We are to prove that a sequence $\sigma_n \to \sigma$ also exists such that $\sigma_n \times \mu_n$ satisfies the constraint for every n. To prove this, we construct a proof of contradiction: assume no such sequence can be found. This means

$$\exists \epsilon > 0, K \in \mathbb{N} \quad s.t. \ (K < k) \Rightarrow \Sigma_{\mu_k} \cap B_\epsilon(\sigma) = \emptyset$$

Since we are guaranteed that $\sigma \in \Sigma_\mu$, it suffices to show

$$\lim_{k \to \infty} \Sigma_{\mu_k} = \Sigma_\mu$$

this is left to the reader. (Check that every member in each side, belongs to the other side too.)

f) We will use the Kakutani's fixed point theorem for the best response map.

$$f\Big(\sigma\Big) = \Big\{\sigma' | \sigma_i' \in br(\sigma_{-i})\Big\}$$

It is defined on a non-empty, convex and compact set. Also for every σ, $f(\sigma)$ is also convex and compact. It is also not difficult to show that it has a closed graph. This completes the proof.

[2] Here we have assumed a proper distance on the set of distributions (eg. Total Variation distance). Later when we mention a convergence of sets, we are using the Hausdorff distance or any other equivalent distance.

Problem 5.65

Problem 5.69

We drop the non-negativity condition on a, b, c, d and instead assume $d = -c$.

For (a, a) to be an ESS, the condition is

$$(c < a) \vee \Big((c = a) \wedge (a + b < 0)\Big)$$

Similary for (d, d) to be an ESS it is necessary to have

$$(-c < b) \vee \Big((b + c = 0) \wedge (a + b < 0)\Big)$$

Finally for the case $p \equiv \frac{c+b}{a+b} \in (0, 1)$, the ESS condition will be

$$p = \frac{b}{a+b} = \frac{c+b}{a+b} \Leftrightarrow c = 0$$

Pset #4

Problem 6.3

GAME C:

The mixed strategy utility is $u(p) = 6p + 2(1-p) = 2 + 4p \to v = 6$

The behavior strategy utility is $u(p) = 6p^2 + 8p(1-p) + 10p(1-p) + 2(1-p)^2 = -10p^2 + 18p + 2 \to v = 10.1$

GAME D, Mixed strategies, player I & II (VN-theorem):

$$u_I(p) = 100 \min(pq, (1-p)(1-q)) \to v_I = v = 25$$

GAME D, Behavior strategies, player I:

$$u_I(p) = 100 \min(pq, (1-p)(1-q)) \to v_I = 25$$

GAME D, Behavior strategies, player II:

$$-u_{II}(p) = 100 \max(p, 1-p) \to v_{II} = 50$$

Problem 6.13

Let us call information sets by different names. A is an information set of another player, Bob, in which he can choose many actions including a_1 and a_2. At some information set B, Alice knows what Bob has played at A. This means "Every path that crosses B, passes through action a_1 of Bob's at A." Then there is another information set of Alice's, called C, such that

i) there is a path going through both B and C.
ii) There are 2 paths that cross C, one of them is generated by Bob playing a_1 and the other by Bob playing a_2.

But this means that the a_2 path crossing C does not pass from B and this further means that Alice does NOT have perfect recall.

Problem 6.15

Again let us use names for info sets. At A, Alice is playing and one of her actions, a leads to the information set C in which charlie will play. At B Bob is playing. If one of his actions b lead to C, then Charlie, at C, does not know who has made the previous move. If there is no action of Bob's (or anyone other than Alice's) that lead to C, then Charlie knows that Alice's action has led to C at all times in the game. This means **The premise can never be fulfilled!** So one could say we have proved this proposition using a vacuous proof.

Problem 6.18

b) Using conditional probabilities one gets the equivalent behavior strategies

$$s_I^B = [\frac{4}{7}(B_1), \frac{3}{7}(T_1)] \times [\frac{3}{4}(B_2), \frac{1}{4}(T_2)] \times [\frac{2}{3}(M_3), \frac{1}{3}(T_3)]$$

$$s_I I^B = [\frac{4}{7}(b_1), \frac{3}{7}(t_1)] \times [\frac{4}{7}(b_2), \frac{3}{7}(t_2)]$$

Problem 6.21

The perfect recall, means the behavior strategies and mixed strategies are equivalent. So a Nash equilibrium must exist. Let us use backward induction. Solving the simple 2×2 strategic form games and replacing them with their Nash values we get another simple game. Iterating this process, solves the game

$$[\frac{2}{5}(T_2), \frac{3}{5}(B_2)] \times [\frac{4}{5}(t_2), \frac{1}{5}(b_2)] \to 10.8$$

Merging this using the random vertex we get a value 7.6.

The bottom game simplifies to

$$[\frac{1}{4}(t_1), \frac{3}{4}(b_1)] \times [\frac{1}{2}(T_3), \frac{1}{2}(B_3)] \to 6$$

This drives I to play B_1.

Problem 6.23

The game matrix looks like

$$\begin{pmatrix} 1 & -2 \\ 1 & 4 \\ 2 & -3 \\ -1 & -1 \end{pmatrix}$$

a) $v = 1.375$

b)

$$u_I(p,q) = \min\Big\{p + (1-p)\big[-(1-q) + 2q\big], p\big[-2q + 4(1-q)\big] + (1-p)\big[-(1-q) - 3q\big]\Big\} \to v_I = 1$$

$$u_{II}(x) = \max\Big\{-2 + 3p,\ 4 - 3p,\ -3 + 5p,\ -1\Big\} \to v_{II} = 1.375 \quad \text{(Expected!)}$$

Pset #5

Problem 7.3

$$\Big(a, \big[p(ce), (1-p)(cf)\big]\Big) \quad \forall p \in [0,1]$$

The equilibria with $p < 1$, are all eliminated in the backward induction method.

Problem 7.4.C

SYNTAX ERROR! The set of actions in all nodes in a common info set should be the same. If we *correct* the game, then it will be a strategic form game

0, 2	3, 2
2, 3	2, 3

The set of perfect equilibria is then

$$\Big(T, \big[y(L), (1-y)(R)\big]\Big) \quad \forall y \in [0, 1/3)$$

$$\Big(\big[x(T), (1-x)(B)\big], \big[1/3(L), 2/3(R)\big]\Big) \quad \forall x \in [0,1]$$

$$\Big(B, \big[y(L), (1-y)(R)\big]\Big) \quad \forall y \in (1/3, 1]$$

Problem 7.21

a) Under the assumption that II plays L, player I feels indifferent about his actions and therefore has no beneficial deviations. For II to have no beneficial deviations, C and R need to be dominated by L under I's mixed strategy.

$$u_{II}\Big([x_1(T), x_2(M), (1-x_1-x_2)(B)], C\Big) \leq u_{II}\Big([x_1(T), x_2(M), (1-x_1-x_2)(B)], L\Big)$$

$$u_{II}\Big([x_1(T), x_2(M), (1-x_1-x_2)(B)], R\Big) \leq u_{II}\Big([x_1(T), x_2(M), (1-x_1-x_2)(B)], L\Big)$$

Which result in the desired conditions.

b) There is no pure equilibrium as the reader can verify. If II randomizes between L and C(R) then T(B) will be the best choice for I and he won't randomize and no equilibrium will follow. It only remains to check the case under which II randomizes between C and R. By checking the best responses of I to II's mixed strategy $[p(C), (1-p)(R)]$ one finds that it is always beneficial for II to shift toward the action L.

c, d) The statements proposed in parts c and d are each other's negations and therefore only one of them can hold. prove that d is true and c is not. d may be rephrased as

There exist a vanishing series of perturbation vectors $(\varepsilon_T, \varepsilon_M, \varepsilon_B, \varepsilon_L, \varepsilon_C, \varepsilon_R)_n$ *inducing Nash equilibria* $\Big(\big[x_1(T), (1-x_1-\varepsilon_B)(M), \varepsilon_B(B)\big], \big[(1-\varepsilon_C-\varepsilon_R)(L), \varepsilon_C(C), \varepsilon_R(R)\big]\Big)$

For the proposed vector to induce an equilibrium, conditions have to be met. I has to be indifferent between M and T:

$$1 + \varepsilon_C + \varepsilon_R = 1 + 2\varepsilon_C - \varepsilon_R$$

Or

$$\varepsilon_C = 2\varepsilon_R$$

I should not prefer B to T or M:

$$\varepsilon_R > 0$$

So far it looks possible to find such perturbation vectors. Now let us look at II. He should not prefer C or R to L:

$$x_1 \geq \frac{1}{3}$$

$$\varepsilon_B \geq 2x_1 - 1$$

The last condition for $x_1 > \frac{1}{2}$ prevents the perturbation vector to approach zero. Therefor it is also necessary to have

$$x_1 \leq .5$$

If all the above conditions above are satisfied then the existence of the perturbation vector is guaranteed. This completes our proof.

e) The answer would be the closure of the set found in (a). But the set is closed and therefore *all the equilibria found in (a) (and no other) are perfect.*

Problem 7.23

No; consider the game

1	1
0	1/2

and the optimal strategy vector (T, R).

Problem 7.36

The one and only perfect equilibrium in the strategic form is found by eliminating the dominated strategies.

$$(B_1 B_2, b)$$

However we will prove that $(T_1 B_2, b)$ is also a perfect equilibrium in the extended form. To do so take the perturbation vector to be $(\varepsilon_{B_1}, \varepsilon_{T_2}, \varepsilon_t)$. The only non-trivial condition to check is $B_1 <_I T_1$ which yields

$$\varepsilon_t < \frac{2\varepsilon_{T_2}}{1 + 2\varepsilon_{T_2}}$$

Which may be satisfied for many perturbation vectors. This completes our proof that $(T_1 B_2, b)$ is indeed a PE.

Problem 7.37

a) By eliminating the dominated strategies in the extended form game we get the unique PE

$$(T_1 B_2, t)$$

b)

1,1	1,1
1,1	1,1
0,2	0,3
0,2	2,0

We propose the equilibrium $(T_1 T_2, t)$ as a potential PE. This is true if there exist εs satisfying

$t > b$:

$$2\varepsilon_{B_1 B_2} > \varepsilon_{B_1 T_2}$$

$T_1 T_2 > B_1 B_2$

$$1 > \varepsilon_b$$

These are easy to satisfy and therefore our conjecture holds.

c) Yes it has and we just found it!

Pset #6

Problem 8.4

To come up with a proof of contradiction, assume the contrary

$$W_i := \sum_\sigma P(\sigma)U_i(\sigma) < \underline{v}_i$$

Then we prove that i can beneficially deviate by playing

$$\sigma'_i := argmax_{\sigma_i} \min_{\sigma_{-i}} U_i(\sigma_i, \sigma_{-i})$$

in contradiction with assumption of equilibrium.

To do so note that

$$\sum_\sigma P(\sigma)U_i(\sigma'_i, \sigma_{-i})$$

$$= \sum_\sigma P(\sigma)U_i(argmax_{\sigma_i} \min_{\sigma_{-i}} U_i(\sigma_i, \sigma_{-i}), \sigma_{-i})$$

$$\geq \sum_\sigma P(\sigma) \max_{\sigma_i} \min_{\sigma_{-i}} U_i(\sigma_i, \sigma_{-i}) = \underline{v}_i > W$$

Problem 8.11

It is easy to check that there are only 3 equilibria with utilities

$$\vec{u}_1 = (4, 9) \quad \vec{u}_2(7, 7) \quad \vec{u}_3(9, 4)$$

We propose a correlated equilibrium as follows

$$P(TL) = \alpha, P(BL) = P(TR) = \frac{1-\alpha}{2}$$

For players not to deviate we need $\alpha \leq \frac{3}{5}$. We take $\alpha \leq \frac{3}{5}$ to raise the (common) utility as high as possible. This results in

$$\vec{u}_C = (7.4, 7.4)$$

To show that it's not in the convex hull of NE's note that

$$(1,1)^T\vec{u}_1 = 13 \quad (1,1)^T\vec{u}_2 = 14 \quad (1,1)^T\vec{u}_3 = 13 \quad (1,1)^T\vec{u}_C = 14.8$$

Problems 8.7 and 8.14

Deviating from a strictly dominated strategy is always beneficial and therefore such a strategy is never played under any (pure, mixed or correlated) equilibrium with positive probability.

Problem 8.21

a) Naming strategies from 1 to n, the strategy n is weakly dominant. As soon as player I(II) randomizes among *any* two of his strategies, player II(I)'s strategy n will be strictly dominant and therefore he will not randomize. This means no mixed equilibrium exists. The diagonal *play*s are easily shown to be pure equilibria and therefore the only equilibria of the game with utilities

$$(x_i, y_i)$$

b) Assume a strategy i is suggested to player I under a correlated equilibrium. Why shouldn't he deviate by playing the weakly dominant strategy n? In fact he will do that unless he knows (with probability 1) that under this suggestion, II will also play i. In other words

$$\forall j \neq i \quad 0 = \mathbb{P}\big[\text{II plays } j | i \text{ is suggested to I}\big] = \frac{P_{ij}}{\sum_k P_{ik}}$$

which is identical to

$$P_{ij} = 0 \quad \forall i \neq j$$

But this means the correlated equilibrium is nothing but a lottery over the diagonal entries hence yielding the utility

$$\vec{u}_C = \sum_i P_{ii}(x_i, y_i)$$

with

$$\sum_i P_{ii} = 1$$

This is what we wanted to prove.

Pset #7

Problem 9.13

a) Connecting two nodes (ω's) if there is at least one player who can not say the difference between them we get

$$1 - 2 - 3 - 4 - 5 - 6$$

$$7 - 8 - 9$$

b) The corresponding connected components are the answers.

$$\omega = 1: \quad 1 - 2 - 3 - 4 - 5 - 6$$

$$\omega = 7: \quad 7 - 8 - 9$$

$$\omega = 8: \quad 7 - 8 - 9$$

c) Under $\omega = 1$ both players know A.

$$K_I A = \{1, 2, 3\}$$

$$K_{II} K_I A = \{\}$$

and therefore does not hold. One can check that $K_I K_{II} A$ also does not hold.

Problem 9.14

The following pictures are scanned from a printed document. The LaTeXcode that generated this document was itself generated by a C code written by the author. Any attempt to do the whole problem by hand is a terrible waste of time!

9.14 a)

00:00	00:01	00:04	00:07	00:10	00:11	00:14	00:17	00:40	00:41	00:44
00:47	00:50	00:51	00:54	00:57	01:00	01:01	01:04	01:07	01:10	01:11
01:14	01:17	01:40	01:41	01:44	01:47	01:50	01:51	01:54	01:57	04:00
04:01	04:04	04:07	04:10	04:11	04:14	04:17	04:40	04:41	04:44	04:47
04:50	04:51	04:54	04:57	07:00	07:01	07:04	07:07	07:10	07:11	07:14
07:17	07:40	07:41	07:44	07:47	07:50	07:51	07:54	07:57	10:00	10:01
10:04	10:07	10:10	10:11	10:14	10:17	10:40	10:41	10:44	10:47	10:50
10:51	10:54	10:57	11:00	11:01	11:04	11:07	11:10	11:11	11:14	11:17
11:40	11:41	11:44	11:47	11:50	11:51	11:54	11:57	14:00	14:01	14:04
14:07	14:10	14:11	14:14	14:17	14:40	14:41	14:44	14:47	14:50	14:51
14:54	14:57	17:00	17:01	17:04	17:07	17:10	17:11	17:14	17:17	17:40
17:41	17:44	17:47	17:50	17:51	17:54	17:57	20:00	20:01	20:04	20:07
20:10	20:11	20:14	20:17	20:40	20:41	20:44	20:47	20:50	20:51	20:54
20:57	21:00	21:01	21:04	21:07	21:10	21:11	21:14	21:17	21:40	21:41
21:44	21:47	21:50	21:51	21:54	21:57					

b)

00:00	00:02	00:04	00:10	00:12	00:14	00:20	00:22	00:24	00:40	00:42
00:44	02:00	02:02	02:04	02:10	02:12	02:14	02:20	02:22	02:24	02:40
02:42	02:44	04:00	04:02	04:04	04:10	04:12	04:14	04:20	04:22	04:24
04:40	04:42	04:44	10:00	10:02	10:04	10:10	10:12	10:14	10:20	10:22
10:24	10:40	10:42	10:44	12:00	12:02	12:04	12:10	12:12	12:14	12:20
12:22	12:24	12:40	12:42	12:44	14:00	14:02	14:04	14:10	14:12	14:14
14:20	14:22	14:24	14:40	14:42	14:44	20:00	20:02	20:04	20:10	20:12
20:14	20:20	20:22	20:24	20:40	20:42	20:44	21:00	21:02	21:04	21:10
21:12	21:14	21:20	21:22	21:24	21:40	21:42	21:44	22:00	22:02	22:04
22:10	22:12	22:14	22:20	22:22	22:24	22:40	22:42	22:44	23:00	23:02
23:04	23:10	23:12	23:14	23:20	23:22	23:24	23:40	23:42	23:44	

c)

00:00	00:04	00:10	00:14	00:40	00:44	04:00	04:04	04:10	04:14	04:40
04:44	10:00	10:04	10:10	10:14	10:40	10:44	14:00	14:04	14:10	14:14
14:40	14:44	20:00	20:04	20:10	20:14	20:40	20:44	21:00	21:04	21:10
21:14	21:40	21:44	22:00	22:04	22:10	22:14	22:40	22:44	23:00	23:04
23:10	23:14	23:40	23:44							

d)

00:00	00:01	00:04	00:07	00:10	00:11	00:14	00:17	00:40	00:41	00:44
00:47	01:00	01:01	01:04	01:07	01:10	01:11	01:14	01:17	01:40	01:41
01:44	01:47	04:00	04:01	04:04	04:07	04:10	04:11	04:14	04:17	04:40
04:41	04:44	04:47	07:00	07:01	07:04	07:07	07:10	07:11	07:14	07:17
07:40	07:41	07:44	07:47	10:00	10:01	10:04	10:07	10:10	10:11	10:14
10:17	10:40	10:41	10:44	10:47	11:00	11:01	11:04	11:07	11:10	11:11
11:14	11:17	11:40	11:41	11:44	11:47	14:00	14:01	14:04	14:07	14:10
14:11	14:14	14:17	14:40	14:41	14:44	14:47	17:00	17:01	17:04	17:07
17:10	17:11	17:14	17:17	17:40	17:41	17:44	17:47	20:00	20:01	20:04
20:07	20:10	20:11	20:14	20:17	20:40	20:41	20:44	20:47	21:00	21:01
21:04	21:07	21:10	21:11	21:14	21:17	21:40	21:41	21:44	21:47	

e)

00:00	00:04	00:10	00:14	00:40	00:44	04:00	04:04	04:10	04:14	04:40
04:44	10:00	10:04	10:10	10:14	10:40	10:44	14:00	14:04	14:10	14:14
14:40	14:44	20:00	20:04	20:10	20:14	20:40	20:44	21:00	21:04	21:10
21:14	21:40	21:44								

f)

$$\mathcal{A} = (N, Y, (\mathcal{F}_i)_{i\in N}, s)$$

With N={William, Dan}, $Y = \{\omega_0, \omega_1, \omega_2, \omega_3, \omega_4, \ldots, \omega_{1440}\}$, $\mathcal{F}_i$ being the partition due to each player's info. set and s the function mapping ω_m to m minutes from midnight. There are clearly 1440=24×60 states of nature and world.

Problem 9.19

a) The first answer is NO; we need to provide a counter example:

$$Y = \{1, 2, 3\}$$

$$\mathcal{F}_A = \{Y\}$$

$$\mathcal{F}_S = \{\{1\}, \{2\}, \{3\}\}$$

Now the event $E_S =$"Sally knows the state of the world" is

$$E_S = \{\omega | F_S(\omega) = \{\omega\}\} = Y$$

and therefore Andrew knows E_S. Although the event $E_A =$"Andrew knows the state of the world" is

$$E_A = \{\omega | F_A(\omega) = \{\omega\}\} = \{\}$$

And therefore never holds. The second answer is AFFIRMATIVE. The reader who seeks exercise should do the following

-Prove $K_S E_S = E_S$

-Prove $K_S K_A E_S = K_A E_S$

-Explain why this implies the result.

The solution we provide here will use the connected component method. Let $C(\omega)$ denote the connected component. We need to show

$$C(\omega) \subseteq K_A E_S$$

Note that to compute a connected component one can first draw the edges due to player I's partitioning. Then proceed by adding player II's and go on until the process terminates. First consider the edges due to Andrew's partitioning. These edges connect ω to vertices only inside E_S (why?). Then consider the edges due to Sally's partitioning. Such edges do not exist since all Sally's partitions contain only one member. This completes our computation

$$C(\omega) = F_A(\omega)$$

$C(\omega) \subseteq K_A E_S$ is then followed immediately by the premise.

b) The premise may be written as $K_A E_S = Y$. Since

$$K_A E_S \subseteq E_S$$

, $E_S = Y$ is obtained. This means YES, YES.

c) same as (a)? [3]

Problem 9.28

We only prove (b) since $\Big((b) \Leftarrow (a)\Big)$ is evident.

It is always possible to describe a situation of incomplete information with common priors using a matrix. The entries are events (not numbers). I knows what row has happened and II knows the

[3]Do you know the difference between the state of nature and the state of the world? Then it would be very kind of you to contact the author and inform him.

column. Therefore the matrix should have $|\mathcal{F}_I|$ rows and $|\mathcal{F}_{II}|$ columns. Denoting this matrix with Ω we have

$$\Omega_{ij} = F_{I,i} \cap F_{II,j}$$

This is the finest (highest resolution) partition of world states that I and II can possibly achieve if they cooperate and therefore we may safely neglect the even finer structures. We also introduce the matrix of common priors

$$P_{ij} \equiv \mathbb{P}[\Omega_{ij}] \geq 0 \qquad \sum_{ij} P_{ij} = 1$$

Finally, for some event A define the matrix

$$A_{ij} \equiv \mathbb{P}[A|\Omega_{ij}]$$

When I announces the probability he ascribes to event A, all rows not satisfying

$$\mathbb{P}[A|R] = p_{announced}$$

may be eliminated. This updates Ω and the common prior matrix P. And the problem gets repeated. If two consecutive announcements are common knowledge (hence not making Ω any smaller), then Aumann's agreement theorem applies and the players reach agreement. Otherwise, Ω gets smaller and eventually shrinks to a single event Ω' and both players agree. There are at most $N_I + N_{II} \leq 2|Y|$ announcements before this point.

Problem 9.39

a)

$$\mathcal{S}_I = \{T_1T_2, T_1B_2, B_1T_2, B_1B_2\}$$
$$\mathcal{S}_{II} = \{L_1L_2, L_1R_2, R_1L_2, R_1R_2\}$$

b)

$$U = \begin{pmatrix} 7.6 & 8.8 & 6.2 & 7.4 \\ 7 & 9.1 & 1 & 3.1 \\ 8.8 & 13.6 & 14.6 & 19.4 \\ 8.2 & 13.9 & 10.2 & 15.1 \end{pmatrix}$$

c) Eliminating the first two row strategies yields

$$U' = \begin{pmatrix} 8.8 & 13.6 & 14.6 & 19.4 \\ 8.2 & 13.9 & 10.2 & 15.1 \end{pmatrix}$$

Now the last three columns may also be eliminated.

$$U" = \begin{pmatrix} 8.8 \\ 8.2 \end{pmatrix}$$

And finally

$$v = (8.8)$$

The corresponding strategies are therefore

$$(L_1L_2, B_1T_2)$$

Problem 12.6

Let's say Brent has a private value w and bids a bid of value B, then (neglecting the probability of a tie) his utility will be

$$U_B = (w - B)\mathbb{P}\big[B > \frac{V + V^2}{3}\big].$$

Solving a quadratic equation yields

$$U_B = \frac{1}{2}(w - B)\Big(\sqrt{3B + \frac{1}{4}} - \frac{1}{2}\Big)$$

a) The optimal bidding strategy maximizes U_B, the first order condition leads to a quadratic equation which in turn gives the result

$$B^*(w) = \frac{2(9w - 1) + \sqrt{4(1 - 9w)^2 - 108w(3w - 1)}}{54}$$

b) One would get the result by substituting B^* into the expression for utility

$$U_B^*(w) = \frac{1}{2}(w - B^*)\frac{\sqrt{3B^* + \frac{1}{4}} - \frac{1}{2}}{2}$$

Problem 12.7

We assume a lottery is performed in the case of a draw.

a) **P1**: There is no equilibrium under which at least one player bids more than 2. (i.e. 3, 4, etc.)

Proof of P1: Suppose there is one such equilibrium and I bids some number b greater than 2 with positive probability. If there is a positive probability that I wins the auction, then he loses some money no matter what his private value was and therefore he is better off bidding 0. If there is no probability that I wins, this means II is always bidding greater than b and winning therefore II is better off bidding 0.

P2: $\beta(V) = 2$ is not an equilibrium strategy.

Proof of P2: It just isn't! (Convince yourself)

P3: There is no equilibrium under which a player bids 2 with positive probability.

Proof of P3: Suppose not, say I is bidding 2 sometimes. If his private value is less than 2, he is losing money with positive probability (note that an all 2 strategy is absent based on **P2**) and therefore he is better off bidding 0. If his private value is 2, he is neither gaining nor losing anything while he can gain some utility by bidding 1 (why?).

P4: $\beta(0) = 0$ in all equilibria.

Proof of P4: Again, assume that this is not the case. Since all of the bids are either 0 or 1, one would win the auction and has to pay for something he does not like at all when bidding a positive amount.

Corollary: Any candidate strategy can be characterized by a pair of probabilities (P_1, P_2). Such that $\beta(V) = Bernoulli(P_V)$. With $P_0 = 0$.

To optimize, we first consider the $V = 1$ case

$$U = \frac{1}{2}(1 - P_1)\mathbb{P}[B' = 0]$$

which gives

$$P_1 = 0 \Rightarrow \beta_1 = 0$$

for $V = 2$ we have

$$U =$$

Problem 12.9

Problem 12.14

Problem 12.21

Problem 12.35

Problem 12.36

Problem 12.40

Problem 12.41

Problem 12.44

Pset #8

Problem 13.8

We denote the value in the repeated game with V. Clearly $V \geq v$ since the player can guarantee $u \geq v$ in every game and therefore $\langle u \rangle \geq v$ in the repeated game using his mM strategy. On the other hand he can *not* guarantee any more, since in every round a punishing strategy played by other players implies $u \leq v \Rightarrow \langle u \rangle \leq v \Rightarrow V \leq v$. This completes our proof.

Problem 13.11

a)

$$\gamma = \frac{1}{6}\Big\{\gamma(B,R) + \gamma(T,R) + \gamma(B,L) + \gamma(T,R) + \gamma(B,R) + \gamma(T,L)\Big\}$$
$$= \frac{1}{6}\Big\{(0,2) + (4,0) + (1,-1) + (4,0) + (0,2) + (-1,3)\Big\} = (4/3,1)$$

b) The game goes as

$$(B,L) \to (T,\sigma) \to (B,R) \to (T,\sigma) \to \cdots$$

This implies

$$u = \frac{1}{2}\big[u(T,\sigma) + u(B,R)\big]$$
$$\frac{1}{2}\big\{\frac{1}{4}u(T,L) + \frac{3}{4}u(T,R) + u(B,R)\big\}$$
$$\frac{1}{2}\big\{(-\frac{1}{4},\frac{3}{4}) + (3,0) + (0,2)\big\} = (\frac{11}{8},\frac{11}{8})$$

c)

$$\sigma_0 = [2/3(T), 1/3(B)]$$

The game is essentially a Markov chain with two states L, R. And

$$P_{R\to L} = P_{L\to L} = \frac{1}{4}$$

This means the limit distribution (strategy) is obtained after one iteration (one round of play).

$$\vec{u} = \vec{u}\Big([2/3(T), 1/3(B)], [3/4(R), 1/4(L)]\Big) = \frac{1}{12}(23,11)$$

Problem 13.14

An iterated ZSG game is a ZSG itself. Therefore at equilibrium, the optimal strategies are played. From **13.8** we have $V = v$. This means every player should play the strategy yielding his optimal utility in each round. This strategy is unique by assumption ($\vec{x}$) and therefore is the unique equilibrium for the iterated game.

Problem 13.16

To compute all the utilities, we write

$$\vec{u}_a = \vec{u}\Big([1/2(T), 1/2(B)], L\Big) = (5.5, 4.5) \Rightarrow u_{II} = 4.5$$

$$\vec{u}_b = \vec{u}\Big([3/4(T), 1/4(B)], R\Big) = (1.5, 5.25) \Rightarrow u_{II} = 5.25$$

$$\vec{u}_c = \vec{u}_d = \frac{1}{2}\Big\{\vec{u}\Big([1/2(T), 1/2(B)], R\Big) + \vec{u}\Big([3/4(T), 1/4(B)], L\Big)\Big\} = (5.5, 7.25) \Rightarrow u_{II} = 7.25$$

This clearly means "c or d" is the answer.

Problem 13.19

Since

$$\vec{u} = \frac{1}{3}\big(\vec{u}(T, L) + \vec{u}(T, R) + \vec{u}(B, R)\big)$$

The following "Grim Trigger" works

I:
While II has not defected play $(L, R, R, L, R, R, L, R, R, \cdots)$ then play B forever.
II:
While I has not defected play $(T, T, B, T, T, B, T, T, B, \cdots)$ then play L forever.

Problem 13.23

For every positive whole number k, we provide an approximation for the utility $\vec{u}$ that is achieved after playing k rounds. Assume

$$\vec{u} = \sum_i \theta_i \vec{u}_i$$

In which $\vec{u}_i$ are pure utilities for the one-round game and θ_i's form a probability vector. To approximate let

$$k\vec{u} \approx \sum_i k_i \vec{u}_i, \qquad k_i \in \mathbb{N}, \quad \sum_i k_i = k$$

It suffices to show that this approximation becomes exact in the limit $k \to \infty$. To do so, first approximate each θ_i with the rational $\frac{\hat{k}_i}{k}$ such that

$$|\theta_i - \frac{\hat{k}_i}{k}| \leq \frac{1}{2k}$$

Then to make $\sum_i k_i = k$ hold, write

$$|\sum_i \frac{\hat{k}_i}{k} - 1| = |\sum_i (\frac{\hat{k}_i}{k} - \theta_i)| \leq \sum_i |\frac{\hat{k}_i}{k} - \theta_i| \leq \frac{|S|}{2k}$$

with $|S|$ denoting the number of all possible plays of the one-round game. This implies

$$|\sum_i \hat{k}_i - k| \leq \frac{|S|}{2}$$

Therefore it is enough to change each $\hat{k}_i$ by at most 1/2 to get to k_i satisfying $\sum_i k_i = k$. This means

$$|\frac{k_i}{k} - \theta_i| \leq \frac{1}{k}$$

Furthermore

$$|\vec{u}_k - \vec{u}| = |\sum_i (\frac{k_i}{k} - \theta_i)\vec{u}_i| \leq \sum_i |(\frac{k_i}{k} - \theta_i)\vec{u}_i|$$
$$\leq \frac{1}{k}\sum_i |\vec{u}_i| \leq \frac{|S|U}{k}$$

With $U = \max|\vec{u}_i|$. This clearly approaches zero as k goes to infinity. Therefore the utility achieved from following these strategies, $\hat{\vec{u}}$ converges to $\vec{u}$. These also induce equilibrium plays since any defection leads to $\vec{v} \leq \vec{u}$. (Every $\vec{u} \in F \cap V$ yields higher utilities than $\vec{v}$)

Problem 13.32

a)

$$\vec{\gamma} = \frac{1-\lambda}{1-\lambda^6}\Big\{(0,2) + \lambda(0,4) + \lambda^2(1,-1) + \lambda^3(4,0) + \lambda^4(0,2) + \lambda^5(-1,3)\Big\}$$
$$= \frac{1-\lambda}{1-\lambda^6}\big(4\lambda + \lambda^2 + 4\lambda^3 - \lambda^5, 2 - \lambda^2 + 2\lambda^4 + 3\lambda^5\big)$$

b)

$$\vec{\gamma} = (1-\lambda)\Big\{\vec{u}(B,L) + \frac{\lambda}{1-\lambda^2}\big[\vec{u}(T,\sigma) + \lambda\vec{u}(B,R)\big]\Big\}$$
$$= (1-\lambda)\Big\{(1,-1) + \frac{\lambda}{1-\lambda^2}\big[\frac{1}{4}(-1,3)\frac{3}{4}(4,0)\lambda(0,2)\big]\Big\}$$

c)

$$\vec{\gamma} = (1-\lambda)\Big\{\vec{u}(\sigma_0, L) + \frac{\lambda}{1-\lambda}\vec{u}\big(\sigma_0, [\frac{1}{4}(L), \frac{3}{4}(R)]\big)\Big\}$$
$$= (1-\lambda)\big[(-\frac{1}{3}, \frac{5}{3}) + \frac{\lambda}{1-\lambda}\frac{1}{12}(23,11)\big]$$

Problem 13.38

a) Suppose each player has $|S_i|$ actions available in the base game. Then

$$\text{Ans.} = |S_i|^{\prod_j |S_j|^k}$$

b) Yes; A "Grimm Trigger" strategy remembers any defect forever hence needing an infinite memory. However if each player has more than one action available and the punishment strategy differs from the strategy before defection, then a player only needs to look at the game played last round to see if any defection has happened or not.

c,d) $(3,3)$ is obtained only when (C,C) is played on all stages. $(T < \infty)$ But in the last round there is no incentive to play C therefore $(3,3)$ is not an equilibrium payoff. This provides an induction proof for "For any $T < \infty$, (3,3) is NOT an equilibrium."

Problem 13.52

Problem 15.9

Dividing I's utility by a and II's utility by b, we get

$$\Big(\frac{\phi_1}{a}, \frac{\phi_2}{b}\Big) = \phi(S', (0,0)), \quad S' = \text{con}\big\{(0,0), (1,0), (0,1)\big\}$$

Using symmetry this is

$$\phi(S', (0,0)) = (\lambda, \lambda)$$

Finally, using efficiency we get $\lambda = \frac{1}{2}$ which proves the desired result.

Problem 15.12

$\phi = d$ definitely satisfies individual rationality. It also satisfies efficiency since in this case $d \in PO(S)$ is guaranteed.

Problem 15.19

a) Without loss of generality assume $\vec{d} = \vec{0}$. Now since S is a compact set, any continuous function, such as $|x_1 - x_2|$ attains its minimum value over S. It remains to show the uniqueness of this point. To construct a proof of contradiction, let $x \in S$ and $x' \in S$ both yield the minimal value for our function. we have

$$|x_1 - x_2| = \min_{s \in S} |s_1 - s_2|$$

$$|x_1' - x_2'| = \min_{s \in S} |s_1 - s_2|$$

Without loss of generality one may assume $x_1 > x_2$. Now if $x_1' > x_2'$ also holds, the two points are either equal (contradiction) or one of them is strictly greater than the other and therefore one is inefficient (contradiction). The case $x_1' < x_2'$ guarantees the existence of a point $x(\lambda) = \lambda x + (1-\lambda)x'$ such that $x_1(\lambda) = x_2(\lambda)$. This violates the minimal value property and is therefore a contradiction.

b) The suggested solution concept clearly satisfies symmetry, efficiency and independence of irrelevant alternatives. We only need to show that it also satisfies covariance under positive affine transformations. The way we have defined the cost function ($C = |(x_1 - d_1) - (x_2 - d_2)|$), implies a covariance under translations and scale-transformations for both x^* and the Y set. This two accordingly result in similar covariances for the solution concept.

Problem 15.26

15.15) If I and II pay taxes of $\alpha\%$ and $\beta\%$ respectively, the money division will go

$$\Big(\frac{2000(1-\alpha)}{2-\alpha-\beta}, \frac{2000(1-\beta)}{2-\alpha-\beta}\Big)$$

And the utilities

$$\Big(\frac{2000(1-\alpha)^2}{2-\alpha-\beta}, \frac{2000(1-\beta)^2}{2-\alpha-\beta}\Big)$$

15.16) The available set (or at least the boundary) is

$$S = \big\{(x-16, \sqrt{2000-x}-7) | x \in [16, 1951]\big\}$$

Solving the equation

$$\frac{x-16}{\sqrt{2000-x}-7} = \frac{1935}{\sqrt{1984}-7}$$

and finding the closest whole number, we get

$$x = 1154$$

Problem 15.28

Extra Problems

* *The expression in the parentheses denotes the subject of each problem.*

Dr. Ejtehadi's Game. (Utility Theory, Single Player Decision Making) In a single player game, Alice (the player) is shown a random number x from $U(0,1)$ in each stage. She has to decide in each stage to keep the number (which will be regarded as her final score), or try again. She can only ask for a retry N times.

a) Assume Alice's utility to be the same as his score. $u(x) = x$. Describe her set of strategies, determine the optimal strategy and find her utility under this optimal utility. Note that utility of a given strategy is a pure number and not a random variable.

b) Assume she is a risk-seeking player. ($u(0) = 0$, $u(1) = 1$, $u''(x) > 0$). Do part a again and compare her optimal utility.

Multiplayer Dr. Ejtehadi's Game. (Extended Form Games, Nash Equilibria) Alice is now playing the previous game against Bob. Bob picks N different numbers from $U(0,1)$. He then represents them to Alice in his arbitrary order. Alice can keep the number she sees in each stage and finish the game or discard it and wait for the next number. She will score as high as the number she gets to keep. Bob's utility is minus Alice's utility and they are both assumed to be Risk-Neutral.

a) For $N = 2$, Describe the set of strategies for each player and solve the game (Find a pure Nash equilibrium).

b) Describe the set of strategies for $N = 3$. Use diagrams if necessary.

Stability Conditions. (Pure Nash Equilibria) Suppose N player are playing a game repeatedly. Each player has a continuum of strategies available. i^{th} Player's action is x_i and his utility u_i. $\vec{x} = \vec{0}$ is a Nash equilibrium. In each round each player chooses the action which is a best response to other players' previous actions.

a) Find the conditions on $\vec{u}(\vec{x})$ for $\vec{x} = \vec{0}$ to be a Nash equilibrium. (Assume smoothness in all parts of this problem)

b) Find stability conditions. Verify that it is invariant under positive affine transformations of each player's utility.

Ethnic Segregation. (Nash equilibria) In a country there are two races, black and white, and two cities A and B. The population of both races are assumed to be a thousand and the capacity of each city is also a thousand. A white (black) person enjoys a population with a fraction x of white (black) people some amount $u(x)$.

$$u(x) = \begin{cases} 2x & : 0 \leq x \leq \frac{1}{2} \\ \frac{3}{2} - x & : \frac{1}{2} \leq x \leq 1 \end{cases}$$

Show that there are two segregated and mixed Nash equilibria. Then reason that the segregated equilibrium is the only one that happens.

An Identity. (Zero sum games) Consider a zero sum game in which there is a fully randomized Nash equilibrium i.e. one under which all actions are played with nonzero probability. The utilities are given by matrix U_{ij}. Show that the value of the game is given by

$$v(U) = \frac{1}{U_{ij}^{-1}}$$

for invertible matrices.

Pirates. (Extended Form Games, Backward Induction) Click on the bold word!

Duel(Backward Induction) Alice and Bob are drawing guns at each other while galloping towards each other. Alice (Bob) will hit the target from a distance x with some probability $P_A(x)$ ($P_B(x)$) which is a perfectly known functions to both of them. They each have only one bullet left, hence missing a shot means they will be shot! Show that they will both fire their guns at the first distance x which satisfies

$$P_A(x) + P_B(x) = 1$$

Bargaining. (Backward Induction, Repeated Games) Consider a 1$ bill on a table first discovered by Alice. Later, Bob finds out and claims a share of the bill. Alice offers a share $s \in [0, 1]$. If Bob accepts the share, the game ends but if he complains, a debate will begin and the owner of bill will arrive! (which is worst for both of them)

a) Solve this game.

Alice (Bob) thinks that the owner's arrival time, is an exponential variate such that after each negotiation, he will arrive with some probability $1 - \delta_A$ $(1 - \delta_B)$. the parameters δ_A and δ_B are common knowledge.

b) Solve the game and show the following

i) Alice has advantages.
ii) The player with higher δ parameter is in a better position.
iii) In the limit $\delta \to 1$ for both players, They each win half the money.

Wars of Attrition. (Repeated Games) a) Solve the symmetric game

$$\begin{pmatrix} -c, -c & 1, 0 \\ 0, 1 & 0, 0 \end{pmatrix}$$

From now on assume that the mixed equilibrium is realized.

b) Assume the game gets repeated if the top left $(-c, -c)$ situation happens. Find the expected utilities.

c) In the limit $c \to 0$ Find the distribution (P.D.F.) of utilities.

A Mafia Game (Repeated Games) Consider a game with 4 players. In the beginning of each game one of the players is randomly chosen to be the "Mafia agent" without the others knowing. During the game, players discuss about who might be the Mafia agent and then, based on their discussions, "Kill" (not literally of course) two of the players. (The choice is made via voting) If the Mafia agent is not among the two, then he wins and the other three players win other wise, the Mafia agent wins and the three other players (including the "dead" one) win.

a) Assume that all players play "honestly" and reveal their identities in the beginning of the game. Compute the expected utility for each player.

b) Assume that the players are perfect liars and are therefore capable of perfectly hiding their identities if they are a Mafia agent. As a result, the mutual information between the choice of "executed" players and the identity of the players is zero. Compute the identity of each player if they all decide to hide their identities.

c) For the infinitely repeated version of the game, construct a Nash equilibrium in which everyone plays honestly.

d) Discuss why this equilibrium is never achieved in real life.

Voting (Nash equilibria) To simplify things, assume that in a voting session, every candidate is characterized only by his political standpoints denoted by a real number x. People are distributed over the political spectrum according to a probability measure $\rho(x)$. If someone on x wins the election, every person on x' gains a utility $u(x, x') \equiv -(x - x')^2$. The candidates belong to a finite subset of $\mathbb{R}$ denoted by C. (For example $C = \{\pm 1\}$). Consider only the set of pure strategies

$$\Sigma \equiv C^{\mathbb{R}}$$

For some set of candidates C and a distribution ρ find a pure Nash equilibrium in which not every player votes to the candidate with the most similar ideas.

$$\exists x \in \mathbb{R} \;\; s.t. \;\; \sigma(x) \neq \min_{c \in C} |x - c|$$