Electrodynamics III - HW#1

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1 Tangent Field Discontinuity at Dipole Layers

Let us consider the closed path that starts from the top of the dipole surface, moves a tangent distance $\vec{d\ell}$, goes under the surface, moves back the distance $\vec{d\ell}$, and returns to the top, where it started, we have

$$0 = \vec{E}_{above}.\vec{d\ell} + \frac{D}{\varepsilon_0} + \vec{d\ell}.\frac{\vec{\nabla}D}{\varepsilon} - \vec{E}_{under}.\vec{d\ell} - \frac{D}{\varepsilon_0}$$

or, equivalently

$$\vec{d\ell}.\Big(\vec{E}_{above} - \vec{E}_{under} + \frac{\vec{\nabla}D}{\varepsilon_0}\Big) = 0.$$

Since this is required for any tangent vector $\vec{d\ell}$,

$$\boxed{\vec{E}^{||}_{above} - \vec{E}^{||}_{under} = -\frac{\vec{\nabla}D}{\varepsilon_0}}$$

2 Regularization

Consider a bounded, localised charge distribution $\rho(\mathbf{x})$, the potential

$$\phi(\mathbf{x}) = \frac{1}{(n-2)\Omega_{n-1}\varepsilon_0}\int d^n\mathbf{x}'\ \frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^{n-2}}$$

is a smooth function with a bounded Laplacian proportional to the charge density. Is there a way to apply the Laplacian operator without facing demons such as Dirac's delta function?

One way is to pick a normalised (but not necessarily positive) distribution $m(\mathbf{x})$ with finite moments

$$\int m(\mathbf{x})d^n\mathbf{x} = 1; \quad \int m(\mathbf{x})\mathbf{x}^r d^n\mathbf{x} < \infty$$

Then the smooth ϕ may be approximated as the average

$$\phi_a(\mathbf{x}) \equiv a^{-n}\int d^n\mathbf{x}'\ \phi(\mathbf{x}')m\Big(\frac{\mathbf{x}-\mathbf{x}'}{a}\Big)$$

And the Laplacian becomes

$$\nabla^2\phi_a(\mathbf{x}) = \frac{a^{-n}}{(n-2)\Omega_{n-1}\varepsilon_0}\int d^n\mathbf{z}\rho(\mathbf{z})\ \nabla^2_{\mathbf{x}}\int d^n\mathbf{y}\ \frac{m\big(\frac{\mathbf{x}-\mathbf{y}}{a}\big)}{|\mathbf{z}-\mathbf{y}|^{n-2}}$$

$$= \frac{a^{-n}}{(n-2)\Omega_{n-1}\varepsilon_0}\int d^n\mathbf{z}\rho(\mathbf{z})\ a^2\nabla^2_{\mathbf{r}}\int d^n\mathbf{y}\ \frac{m(\mathbf{y})}{|\mathbf{y}+\mathbf{r}/a|^{n-2}}$$

where $\mathbf{r} \equiv \mathbf{z} - \mathbf{x}$. For spherically symmetric m, this further simplifies into

$$\nabla^2\phi_a(\mathbf{x}) = \frac{\Omega_{n-2}a^{-n}}{(n-2)\Omega_{n-1}\varepsilon_0} \int d^n\mathbf{z}\ \rho(\mathbf{z})a^2\nabla^2_{\mathbf{r}} \int_0^\infty dy\ \frac{y^{n-1}m(y)}{\left[y^2 + r^2/a^2\right]^{n/2-1}} \int_{-1}^{+1} du\ \sqrt{\frac{(1-u^2)^{n-3}}{\left(1 + \frac{2yra}{r^2+y^2a^2}u\right)^{n-2}}}$$

In 3D this becomes

$$\nabla^2\phi_a(\mathbf{x}) = \frac{a^{-1}}{2\varepsilon_0} \int d^3\mathbf{z}\ \rho(\mathbf{z})\nabla^2_{\mathbf{r}} \int_0^\infty dy\ ym(y)\Big(1 + ay/r - \big|1 - ay/r\big|\Big)$$

Picking different m values, allows for different regularizations. As an explicit example, consider a spherical averaging m, that is

$$m(y) = \frac{3}{4\pi R^3} \times 1[y \leq R]$$

The Laplacian becomes

$$\begin{aligned}
\nabla^2\phi_a(\mathbf{x}) &= \int d^3\mathbf{z}\ \frac{\rho(\mathbf{z})}{\varepsilon_0}\nabla^2_{\mathbf{r}} \begin{cases} \frac{3}{8\pi aR} - \frac{r^2}{8\pi(aR)^3} & r \leq aR \\ \frac{1}{4\pi r} & r \geq aR \end{cases} \\
&= \frac{-3}{4\pi\varepsilon_0(aR)^3} \int_{|\mathbf{r}|\leq aR} d^3\mathbf{r}\ \rho(\mathbf{x}+\mathbf{r}) \\
&= -\frac{\rho(\mathbf{x})}{\varepsilon_0} + \mathcal{O}(a^2) \quad \blacksquare
\end{aligned}$$

Note that nowhere in our calculation, we faced an improper integral, a delta function, or any other such divergences.

3 Green's Reciprocity and Applications

To prove Green's Reciprocity theorem start by writing

$$\oint_{\partial V} (\sigma\phi' - \sigma'\phi)da = \varepsilon_0 \oint_{\partial V} (\phi'\vec{\nabla}\phi - \phi\vec{\nabla}\phi').\vec{da} = \varepsilon_0 \int_V \vec{\nabla}.\Big(\phi'\vec{\nabla}\phi - \phi\vec{\nabla}\phi'\Big)dV$$

$$= \int_V (\phi\rho' - \phi'\rho)dV$$

which is (equivalent to) what we wanted to prove.

As the first application, consider a set of conductors, the total charge on the ith conductor, denoted by Q_i, is a linear function of the potentials V_i:

$$Q_i = C_{ij}V_j$$

Now consider two cases: in the first one, the ith conductor is held at unit potential while all the others are grounded, in the second case, the i'th conductor is at unit potential and the rest are grounded. The reciprocity theorem asserts that

$$Q_{i'} = Q'_i$$

which implies

$$\boxed{C_{ii'} = C'_i}$$

As another application, consider a unit point chargeat point $\mathbf{x}$ in the presence of grounded conductors. Each conductor will gain a charge $a_i(\mathbf{x})$. As the second case, let each conductor be at some potential V_i while there are no volume charges; this will induce a potential $V_iK_i(\mathbf{x})$ at each point. The reciprocity theorem yields

$$\sum_i V_i\big(a_i(\mathbf{x}) + K_i(\mathbf{x})\big) = 0$$

Since V_i are arbitrary, we find

$$\boxed{a_i(\mathbf{x}) = -K_i(\mathbf{x})}$$

An explicit example, would be the case of a point charge near a spherical conductor; we find that the induced charge is given by

$$Q_{ind.} = -\frac{R}{a}Q$$

where R is the conductor's radius and a is the charge's distance from the center of the sphere.

4 Jackson; 1-17

a) The energy in such a setting is given by

$$W = \frac{1}{2}\sum_i Q_i V_i = \frac{1}{2}\sum_i V_i \sum_j C_{ij} V_j = \frac{1}{2}C_{ii} =: \frac{C}{2}$$

Equivalently, this is given as the volume integral of the energy density

$$\frac{C}{2} = \frac{\varepsilon_0}{2}\int_V dV\ |\vec{\nabla}\phi|^2$$

which is the same as

$$\boxed{C = \varepsilon_0 \int_V dV\ |\vec{\nabla}\phi|^2}$$

b) The true potential, ϕ is that which minimizes the total energy while satisfying the boundary conditions.(This was proved in the class and in the chapter); therefore, any other ansatz potential ψ that satisfies the boundary conditions, gives an upper bound for the true capacitance:

$$\boxed{C \leq \int_V dV\ |\vec{\nabla}\psi|^2}$$

To prove that this is stationary at the true potential, we use the Euler-Lagrange equations for classical field theories

$$0 = \frac{\partial|\vec{\nabla}\psi|^2}{\partial\psi} = \partial_i \frac{\partial|\vec{\nabla}\psi|^2}{\partial\partial_i\psi} = 2\nabla^2\psi$$

To prove that this gives a minimum and not a maximum, consider the potential corresponding to the same boundary conditions but with a few added point charges. The energy for such a potential would be infinitely large and therefore we find that there is no maximum for the variational problem; thereby completing our proof.

5 Jackson 1-18

a) All we need to do is to consider the charges on the surface of S_1 as free charges instead of induced ones, then the problem will consist of all the other conductors held at zero potential and the free charges on the surface of S_1. The solution is (by definition of G) given by

$$\phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\oint_{S_1} da'\ \sigma_1(\mathbf{x}')G(\mathbf{x},\mathbf{x}')$$

The charge-potential formula for the electrostatic energy then yields

$$W = \frac{1}{2}\oint_{S_1} da\ \phi(\mathbf{x})\sigma_1(\mathbf{x})$$

$$\boxed{W = \frac{1}{8\pi\varepsilon_0} \oint_{S_1} da \oint_{S_1} da' \ \sigma_1(\mathbf{x})\sigma_1(\mathbf{x}')G(\mathbf{x},\mathbf{x}')}$$

b) The electrostatic potential may also be written as

$$W = \frac{1}{2}CV^2 = \frac{Q^2}{2C}$$

which means

$$C^{-1} = \frac{2W}{Q^2} = \frac{\oint_{S_1} da \oint_{S_1} da' \ \sigma_1(\mathbf{x})\sigma_1(\mathbf{x}')G(\mathbf{x},\mathbf{x}')}{4\pi\varepsilon_0 \left[\oint_{S_1} da \, \sigma_1(\mathbf{x}) \right]^2}$$

If σ is considered as a generic charge distribution for the functional C^{-1}, then we have

$$\delta C^{-1} = \frac{Q\delta \oint \phi\sigma da - 2\delta Q \oint \phi\sigma da}{Q^3}$$

$$= \frac{1}{Q^2}\left\{ \oiint_{S_1} dada'\sigma(\mathbf{x})G(\mathbf{x},\mathbf{x}')\delta\sigma(\mathbf{x}') - \delta Q \right\}$$

$$= \frac{1}{Q^2} \oiint_{S_1} dada' G(\mathbf{x},\mathbf{x}') \Big[\sigma(\mathbf{x})\delta\sigma(\mathbf{x}') - \delta\sigma(\mathbf{x})\sigma(\mathbf{x}') \Big] = 0$$

Where in the last line we have used the symmetry property of the Green function. Once again, we know that this quantity may be made arbitrarily large (consider a point distribution on the surface S_1) and therefore, the true charge distribution σ_1 yields the minimal value. In other words

$$C \geq \frac{4\pi\varepsilon_0 \left[\oint_{S_1} da \, \sigma(\mathbf{x}) \right]^2}{\oint_{S_1} da \oint_{S_1} da' \, \sigma(\mathbf{x})\sigma(\mathbf{x}')G(\mathbf{x},\mathbf{x}')}$$