Electrodynamics III - HW#2

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1 Max or Min

In the last set of problems, we argued that the action integral is minimized (and not maximized) at the physical potential by comparing the action with that corresponding to the physical potential around a point charge in the volume. Here, we directly prove that the action is convex (has positive definite second derivative) to show that the extremal value is a minimum.

To do this, all we need is to expand the action (1.67) up to second order (it is quadratic, so there will be no higher order terms) and show that the second order term is positive definite.

$$I[\psi+\chi] = I[\psi] + \int_V d^n\mathbf{x}\,(\nabla\psi.\nabla\chi) - \int_V dV\, g\chi - \oint_{\partial V} da\, f\chi + \frac{1}{2}\int_V dV\,|\nabla\chi|^2$$

We need not look any further; the second order term (in χ) is conspicuously positive definite.

2 Jackson 2.14

I use s instead of ρ in cylindrical coordinates.

a) The potential is well defined at the origin and therefore the negative powers are excluded, also, we know that a $\pi/2$ rotation will turn the potential to minus itself and therefore we can also exclude the constant term and all odd integer powers. The potential also has odd symmetry (in φ) which rules out the cosine terms. Finally, the combination of these symmetries, $\varphi \to \frac{\pi}{2} - \varphi$ further restricts the allowed modes

$$\sin(2m\varphi) = \sin\left[2m(\frac{\pi}{2}-\varphi)\right] = \sin(m\pi - 2m\varphi) = (-1)^{m+1}\sin(2m\phi) \;\Rightarrow\; m = 2n+1$$

That is, in general, we have the following form

$$\phi(s,\varphi) = \sum_{n=0}^{\infty} \frac{A_n}{\sqrt{\pi}}\left(\frac{s}{b}\right)^{4n+2}\sin\left[(4n+2)\varphi\right]$$

This convention for the unknown coefficients, makes it easy to use the dot-product rule on the boundary to find the A_n:

$$A_n = \frac{V}{\sqrt{\pi}}\int_0^{2\pi}\sin\left[(4n+2)\varphi\right]\operatorname{sgn}\Big[\sin(2\varphi)\Big]\,d\varphi = \frac{4V}{(4n+2)\sqrt{\pi}}\int_0^{(2n+1)\pi} dx\,\sin x = \frac{4v}{(2n+1)\sqrt{\pi}}$$

Which finally yields

$$\boxed{\phi(s,\varphi) = \frac{4V}{\pi}\sum_{n=0}^{\infty}\left(\frac{s}{b}\right)^{4n+2}\frac{\sin\left[(4n+2)\varphi\right]}{2n+1}}$$

b) We start by acting with operator $s\partial_s$ on the potential

$$s\partial_s\phi(s,\varphi) = \frac{8V}{\pi}\sum_{n=0}^{\infty}\left(\frac{s}{b}\right)^{4n+2}\sin\left[(4n+2)\varphi\right] = \frac{8V}{\pi}\operatorname{Im}\left\{\sum_{n=0}^{\infty}\left(\frac{s}{b}e^{i\varphi}\right)^{4n+2}\right\} = -\frac{4V}{\pi}\operatorname{Im}\Big\{\operatorname{csch}(2w)\Big\}$$

where I have introduced the short-hand $w \equiv \log\left(\frac{s}{b}e^{i\varphi}\right)$. Now if the real field ψ is chosen such that $f(w) = \phi + i\psi$ is analytic, (This is always possible for harmonic functions; Cf. complex analysis) then we have

$$\frac{df}{dw} = \frac{4iV}{\pi}\operatorname{csch}(2w)$$

This integrates as

$$f(w) = C + \frac{2iV}{\pi}\log\tanh w$$

At $w \to -\infty$, this has to have zero real component and therefore we get $C = 2V$. Finally

$$\phi(s,\varphi) = \frac{2V}{\pi}\left[\pi - \arg\left(\frac{z^2-1}{z^2+1}\right)\right]$$

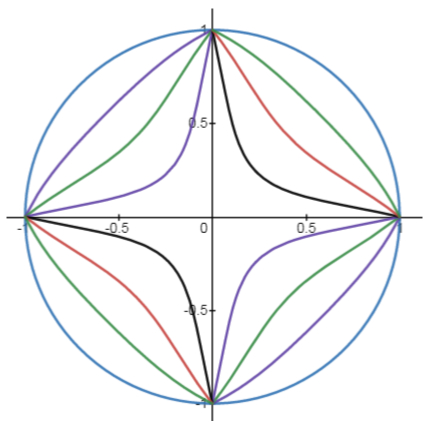
with $z \equiv \frac{s}{b}e^{i\varphi}$. Since ϕ is bounded between $\pm V$, we only need the tangent of the angle inside the brackets, a simple calculation yields

$$\phi = \frac{2V}{\pi}\arctan\left[\frac{2(s/b)^2\sin(2\varphi)}{1-(s/b)^4}\right]$$

c) On desmos.com, we draw the graph corresponding to

$$\frac{4xy}{1-(x^2+y^2)^2} = C$$

for constant C, here is the result.



3 Jackson 2.15

a) At first, it is only clear (from the boundary conditions) that we may write

$$G = \sqrt{2}\sum_{n=1}^{\infty} f_n(y;x',y')\sin(n\pi x)$$

Then the differential equation to satisfy will be

$$\sqrt{2}\sum_{n=1}^{\infty}\sin(n\pi x)\Big(-n^2\pi^2 + \partial_y^2\Big)f_n(y;x',y') = -4\pi\delta(x-x')\delta(y-y')$$

Calculating the dot product with $\sqrt{2}\sin(n\pi x)$ yields

$$\Big(-n^2\pi^2 + \partial_y^2\Big)f_n = -4\sqrt{2}\pi\sin(n\pi x')\delta(y-y')$$

This equation admits a solution in the form

$$f_n = \sqrt{2}\sin(n\pi x')g_n(y;y')$$

which leads to the expansion

$$\boxed{G = 2\sum_n g_n(y;y')\sin(n\pi x)\sin(n\pi x')}$$

and the differential equation becommes

$$\boxed{\Big(-n^2\pi^2 + \partial_y^2\Big)g_n = -4\pi\delta(y-y')}$$

b) In the empty regions ($y \neq y'$) the solution is a linear combination of $\sinh(n\pi y)$ and $\cosh(n\pi y)$; the boundary condition implies

$$g_n(y,y') = \begin{cases} A\sinh(n\pi y) & y \leq y' \\ B\sinh[n\pi(1-y)] & y \geq y' \end{cases}$$

the delta function leads to a slope discontinuity:

$$\partial_y g_n(y'^+;y') - \partial_y g_n(y'^-;y') = -4\pi$$

This, along with the continuity condition, gives us the set of equations

$$\begin{cases} A\sinh(n\pi y') = B\sinh[n\pi(1-y')] \\ A\cosh(n\pi y') + B\cosh[n\pi(1-y')] = \frac{4}{n} \end{cases}$$

which solve as

$$A = \frac{4}{n\sinh(n\pi)}\sinh[n\pi(1-y')]; \quad B = \frac{4}{n\sinh(n\pi)}\sinh(n\pi y')$$

Substituting these back into our formula for G, gives us what we were looking for

$$\boxed{G = \sum_{n=1}^{\infty}\frac{8}{n\sinh(n\pi)}\sin(n\pi x)\sin(n\pi x')\sinh(n\pi y_<)\sinh[n\pi(1-y_>)]}$$

4 Jackson 2.26

a) In general

$$\phi = A + B\log s + C\varphi + \sum_{p\in\mathbb{R}^+}\left\{\Big[D_p^+\big(\frac{s}{a}\big)^p + D_p^-\big(\frac{s}{a}\big)^{-p}\Big]\sin(p\varphi) + \Big[E_p^+\big(\frac{s}{a}\big)^p + E_p^-\big(\frac{s}{a}\big)^{-p}\Big]\cos(p\varphi)\right\}$$

However, the boundary conditions at $\varphi \in \{0,\beta\}$ exclude all the terms except the sin ones; they also quantise the allowed values of n:

$$\phi = \sum_{n=1}^{\infty}\sin\big(\frac{n\pi\varphi}{\beta}\big)\Big[D_n^+\big(\frac{s}{a}\big)^{n\pi/\beta} + D_n^-\big(\frac{s}{a}\big)^{-n\pi/\beta}\Big]$$

Finally, the boundary condition at $s = a$, gives

$$\boxed{\phi = \sum_{n=1}^{\infty}D_n\sin\big(\frac{n\pi\varphi}{\beta}\big)\Big[\big(\frac{s}{a}\big)^{n\pi/\beta} - \big(\frac{s}{a}\big)^{-n\pi/\beta}\Big]}$$

b) Keeping only the $n = 1$ term, the potential is

$$\phi = V\sin\big(\frac{\pi\varphi}{\beta}\big)\Big[\big(\frac{s}{a}\big)^{\pi/\beta} - \big(\frac{s}{a}\big)^{-\pi/\beta}\Big]$$

The electric field follows immediately

$$\boxed{\mathbf{E} = -\frac{\pi V}{\beta a}\Big\{\hat{s}\sin\big(\frac{\pi\varphi}{\beta}\big)\Big[\big(\frac{s}{a}\big)^{\pi/\beta-1} + \big(\frac{s}{a}\big)^{-\pi/\beta-1}\Big] + \hat{\varphi}\cos\big(\frac{\pi\varphi}{\beta}\big)\Big[\big(\frac{s}{a}\big)^{\pi/\beta-1} - \big(\frac{s}{a}\big)^{-\pi/\beta-1}\Big]\Big\}}$$

To find the surface charge densities, all we need is to evaluate the normal component of the electric field at the boundaries and multiply by ε_0. At $\varphi = 0$ and $\varphi = \beta$ this gives

$$\boxed{\sigma_0(s) = \sigma_\beta(s) = -\frac{\pi\varepsilon_0 V}{\beta a}\Big[\big(\frac{s}{a}\big)^{\pi/\beta-1} - \big(\frac{s}{a}\big)^{-\pi/\beta-1}\Big]}$$

similarly, at $s = a$:

$$\boxed{\sigma_a(\varphi) = -\frac{2\pi\varepsilon_0 V}{\beta a}\sin\big(\frac{\pi\varphi}{\beta}\big)}$$

c) For $\beta = \pi$, the field simplifies as

$$\boxed{\mathbf{E} = -\frac{V}{a}\hat{y} + \mathcal{O}\Big(\frac{a^2}{s^2}\Big)}$$

This is similar to the field from a uniform surface charge density of $\sigma \equiv -\varepsilon_0 V/a$

The charge density on the half-cylinder integrates as

$$\boxed{\frac{dQ}{dz} = -\frac{2\varepsilon_0 V}{a}\int_0^\pi a d\varphi\,\sin\varphi\ = -4\varepsilon_0 V = 2\times(2a\sigma)}$$

This is the same as the charge deficiency in other areas

$$\boxed{2\times\int_a^\infty ds\,\big[\sigma - \sigma_0(s)\big] = 2\sigma a^2\int_a^\infty \frac{ds}{s^2} = 2a\sigma}$$

Finally, to visualise the charge density, we draw a diagram (on desmos.com) in which the height (relative to the conductor surfaces) is proportional to the charge density.