Electrodynamics III - HW#3

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November 2021

1 $P_l(\cos\gamma)$ in terms of Y_{lm}s; A proof for Jackson's eq. 3.62

For $\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varphi-\varphi')$, and fixed θ', φ', the $P_l(\cos\gamma)$ is an angular function of θ, φ and may therefore be described as a sum over the Y_{lm} spherical harmonics. Let us start by acting on the function with the angular Laplacian operator

$$\Delta \equiv \frac{1}{\sin\theta}\partial_\theta \sin\theta\partial_\theta + \frac{1}{\sin^2\theta}\partial_\varphi^2$$

we get

$$\Delta P_l(\cos\theta) = P_l''(\cos\gamma)\Big\{\big(\frac{\partial\cos\gamma}{\partial\theta}\big)^2 + \frac{1}{\sin^2\theta}\big(\frac{\partial\cos\gamma}{\partial\varphi}\big)^2\Big\} + P_l'(\cos\gamma)\Big\{\cot\theta\frac{\partial\cos\gamma}{\partial\theta} - \cos\gamma - \frac{\sin\theta'}{\sin\theta}\cos(\varphi-\varphi')\Big\}$$

$$= (1-\cos^2\gamma)P_l''(\cos\gamma) - 2\cos\gamma P_l'(\cos\gamma) = l(l+1)P_l(\cos\gamma)$$

Where in the last line, we have used the Legendre differential equation. This means that $P_l(\cos\gamma)$ is an eigenfunction of the operator Δ with the eigenvalue $l(l+1)$ which implies

$$P_l(\cos\gamma) = \sum_{m=-l}^{+l} A_{lm}(\theta',\varphi')Y_{lm}(\theta,\varphi)$$

where

$$A_{lm}(\theta',\varphi') \equiv \int_{\mathbb{S}^2} d\Omega\, Y_{lm}^*(\Omega)P_l(\cos\gamma)$$

$$= \int_0^\pi d\gamma\sin\gamma\int_0^{2\pi} d\beta\ Y_{l0}^*(\gamma,\beta)\Big[\sqrt{\frac{4\pi}{2l+1}}Y_{lm}^*\big(\theta(\gamma,\beta\,|\,\theta',\varphi'),\varphi(\gamma,\beta\,|\,\theta',\varphi')\big)\Big] = B_{l0}(\theta',\varphi')$$

where B_{lm} satisfy

$$\sqrt{\frac{4\pi}{2l+1}}Y_{lm}^*\big(\theta(\gamma,\beta\,|\,\theta',\varphi'),\varphi(\gamma,\beta\,|\,\theta',\varphi')\big) = \sum_{m=-l}^{+l} B_{lm}(\theta',\varphi')Y_{lm}(\gamma,\beta)$$

At $\gamma = 0$ (at the primed z-axis, or even more explicitly at $\theta = \theta'$ and $\varphi = \varphi'$) all the $m \neq 0$ terms vanish and this gives

$$B_{l0}(\theta',\varphi')\sqrt{\frac{2l+1}{4\pi}} = \sqrt{\frac{4\pi}{2l+1}}Y_{lm}^*(\theta',\varphi')$$

Finally, substituting this back to A_{lm}, we get

$$A_{lm}(\theta',\varphi') = \frac{4\pi}{2l+1}Y_{lm}^*(\theta',\varphi')$$

which completes our proof for the addition theorem:

$$\boxed{P_l(\cos\gamma) = \frac{4\pi}{2l+1}\sum_{m=-l}^{+l} Y_{lm}^*(\theta',\varphi')Y_{lm}(\theta,\varphi)}$$

2 Jackson; 3.2

a) The potential may be written as

$$\phi(r,\theta) = \begin{cases} A_l \left(\frac{r}{R}\right)^l P_l(\cos\theta) & r \le R \\ A_l \left(\frac{R}{r}\right)^{l+1} P_l(\cos\theta) & r \ge R \end{cases}$$

The constants are found by the field discontinuity condition

$$\sigma(\cos\theta) = \varepsilon_0\, E_r\Big|_{R^-}^{R^+} = \varepsilon_0 \sum_l \frac{2l+1}{R} A_l P_l(\cos\theta)$$

Using the orthogonality, we get

$$A_l = \frac{Q}{8\pi\varepsilon_0 R}\int_{-1}^{\cos\alpha} P_l(x)dx$$

Using Jackson's eq. 3.28

$$P_l(x) = \frac{1}{2l+1}\left[P'_{l+1}(x) - P'_{l-1}(x)\right]$$

which holds for $P_{-1}(x) = C \in \mathbb{R}$, we get

$$A_l = \frac{Q}{8\pi\varepsilon_0 R}\frac{P_{l+1}(\cos\alpha) - P_{l-1}(\cos\alpha)}{2l+1}$$

therefore

$$\boxed{\phi(r,\theta) = \frac{Q}{8\pi\varepsilon_0 R}\sum_{l=0}^{\infty}\frac{P_{l+1}(\cos\alpha) - P_{l-1}(\cos\alpha)}{2l+1}P_l(\cos\theta)\begin{cases}(r/R)^l & r \le R \\ (R/r)^{l+1} & r \ge R\end{cases}}$$

For $r \gg R$ and $\alpha = 0$, this is

$$\phi = \frac{Q(1-C)}{8\pi\varepsilon_0 r} + \mathcal{O}(1/r^2)$$

which is the correct answer provided that $\boxed{C = -1}$

b) This is given by

$$\mathbf{E}(O) = -\hat{\mathbf{z}}\lim_{r\to 0^+}\frac{\partial\phi}{\partial r}\Big|_{\theta=0} = \frac{Q\hat{\mathbf{z}}}{24\pi\varepsilon_0 R^2}\left[1 - P_2(\cos\alpha)\right] = \boxed{\frac{Q\sin^2\alpha\ \hat{\mathbf{z}}}{16\pi\varepsilon_0 R^2}}$$

c) The potential:

$$\phi(O) = \frac{Q}{4\pi\varepsilon_0 R}\cos^2(\alpha/2) = \boxed{\frac{Q_{tot.}}{4\pi\varepsilon_0 R}}$$

For small cap, the total charge approaches Q and for large cap, the total charge approaches that of a disk of radius $R(\pi - \alpha)$ with the same surface density.

The field: The expression that we found in the last part for the field, has the same limiting behavior for both large and small caps: For large caps, this is similar to the field generated by a point charge at the south pole of magnitude given by the product of the surface density and the cap area. In the small cap limit, the setting may be regarded as the superposition of the full spherical shell (with no net field) and a cap of negative charge in the north pole; the field will be the same.

3 Jackson; 3.3

a) First, let us find the proportion constant for the surface charge density in terms of the potential. To do this we evaluate the potential at the origin:

$$V = \int_0^R \frac{2\pi\sigma r dr}{4\pi\varepsilon_0 r\sqrt{1-(r/R)^2}} = \frac{\sigma R}{2\varepsilon_0}\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \frac{\pi\sigma R}{4\varepsilon_0} \Rightarrow \sigma = \frac{4\varepsilon_0 V}{\pi R}$$

The potential for $r \geq R$ is given by the Coulomb integral

$$\phi(r,\theta) = \frac{V}{\pi^2 Rr}\int_0^R \frac{r'dr'}{\sqrt{1-(r'/R)^2}}\sum_{l=0}^{\infty}\left(\frac{r'}{r}\right)^l\int_0^{2\pi} d\varphi' P_l(\cos\gamma)$$

$$= \frac{4V}{\pi Rr}\int_0^R \frac{r'dr'}{\sqrt{1-(r'/R)^2}}\sum_{l=0}^{\infty}\left(\frac{r'}{r}\right)^l \frac{1}{2l+1}\int_0^{2\pi} d\varphi' \sum_{m=-l}^{+l} Y^*_{lm}(\frac{\pi}{2},\varphi')Y_{lm}(\theta,\varphi)$$

$$= \frac{2V}{\pi Rr}\int_0^R \frac{r'dr'}{\sqrt{1-(r'/R)^2}}\sum_{l=0}^{\infty}\left(\frac{r'}{r}\right)^l P_l(0)P_l(\cos\theta)$$

$$= \frac{V}{\pi}\sum_{l=0}^{\infty}\left(\frac{R}{r}\right)^{2l+1} P_{2l}(\cos\theta)P_{2l}(0)\int_0^1 \frac{x^l dx}{\sqrt{1-x}}$$

$$= \frac{V}{\pi}\sum_{l=0}^{\infty}\left(\frac{R}{r}\right)^{2l+1} P_{2l}(\cos\theta)\frac{(-)^l\binom{2l}{l}}{4^l}\int_0^{\pi} dx \sin^{2l+1}(x)$$

where in the last line we have made use of the 'generating function' formula for Legendre polynomials to evaluate $P_{2l}(0)$. Now if we define

$$J_n \equiv \int_0^{\pi} dx \sin^{2n+1}(x)$$

then an integration by parts gives (for $n > 0$)

$$J_n = \frac{2n}{2n+1}J_{n-1}; \quad J_0 = 2. \implies J_n = \frac{2}{2n+1}\frac{4^n}{\binom{2n}{n}}$$

This proves what we wanted

$$\boxed{\phi_{out.}(r,\theta) = \frac{2V}{\pi}\sum_{l=0}^{\infty}\frac{(-)^l}{2l+1}\left(\frac{R}{r}\right)^{2l+1} P_{2l}(\cos\theta)}$$

b) Due to the symmetry of the problem, the potential is the same in both the northern and the southern hemisphere. We only find it in the northern hemisphere.

$$\phi = \sum_{l=0}^{\infty} A_l P_l(\cos\theta)\left(\frac{r}{R}\right)^l$$

The Dirichlet boundary condition at $\theta = \frac{\pi}{2}$ excludes all the even ls except the $l = 0$ term:

$$\phi = V + \sum_{l=0}^{\infty} B_l P_{2l+1}(\cos\theta)\left(\frac{r}{R}\right)^{2l+1}$$

To find B_l, thanks to Jackson's hint, we can use either the Neumann boundary condition at $\theta = \frac{\pi}{2}$:

$$\frac{2V/\pi}{\sqrt{1-x}} = -\sum_{l=0}^{\infty} B_l P'_{2l+1}(0)x^l \;\Rightarrow\; B_l = \frac{-2V}{\pi}\frac{(2l)!}{4^l(l!)^2}\frac{1}{P'_{2l+1}(0)}$$

or the Dirichlet boundary condition found from the previous part at $r = R$:

$$V + \sum_{l=0}^{\infty} B_l P_{2l+1}(|x|) = \frac{2V}{\pi} \sum_{l=0}^{\infty} \frac{(-)^l}{2l+1} P_{2l}(x) \;\Rightarrow\; \sum_{n=0}^{\infty} B_n \int_0^1 P_{2n+1}(x) P_{2l}(x) dx \;= V\Big[\frac{(-)^l(2/\pi)}{(2l+1)(4l+1)} - \delta_{l0}\Big]$$

Clearly, the Neumann bc is much more convenient to use. It remains to find the coefficients $P'_{2l+1}(0)$. Using Jackson's eq. 3.28 and the initial condition $P'_1(0) = 1$, we get

$$P'_{2l+1}(0) = \sum_{n=0}^{l} (4n+1)\big(\frac{-1}{4}\big)^n \binom{2n}{n} = \frac{1}{2}\Big(\frac{-1}{4}\Big)^l (l+1) \binom{2l+2}{l+1}$$

Which gives

$$B_l = \frac{-2V}{\pi} \frac{(-)^l}{2l+1}$$

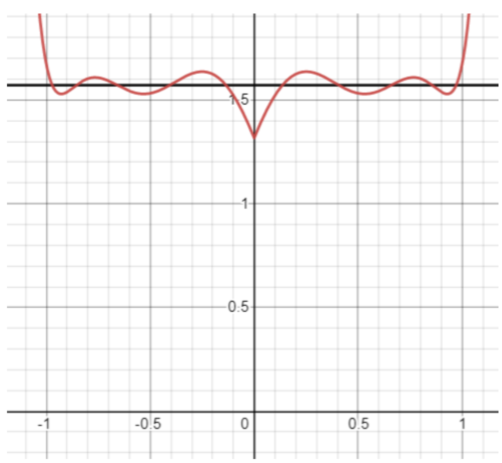
therefore, for $r \leq R$ we get

$$\boxed{\phi_{ins.}(r, \theta) = V - \frac{2V}{\pi} \sum_{l=0}^{\infty} \frac{(-)^l}{2l+1} P_{2l+1}(|\cos\theta|) \Big(\frac{r}{R}\Big)^{2l+1}}$$

As a bonus, in regard for Jackson's hint about the charge distribution on the disk, we get an identity:

$$\boxed{\frac{\pi}{2} = \sum_{n=0}^{\infty} \frac{(-)^n}{2n+1} \Big[P_{2n}(x) + P_{2n+1}(|x|)\Big]}$$

A plot for the first 5 terms in this series is shown below



c)

$$C = \frac{1}{V} \int_0^R 2\pi r dr \, \frac{4\varepsilon_0 V / \pi R}{\sqrt{1 - (r/R)^2}} = 8\varepsilon_0 R \int_0^1 \frac{x dx}{\sqrt{1 - x^2}} = \boxed{8\varepsilon_0 R}$$

4 Jackson; 3.5

We find the solution using two different approaches; the uniqueness theorem guarantees their equivalence.

a) First, we may use the Green function method, eq. 1.42 to get

$$\phi(r,\theta,\varphi) = \frac{-a^2}{4\pi}\int_{\mathbb{S}^2} d\Omega' V(\Omega')\frac{\partial G_D(r,\theta,\varphi;r',\theta',\varphi')}{\partial r'}$$

Using the method of images, it is easy to compute the G_D and its derivative as

$$\frac{\partial G_D(r,\theta,\varphi;r',\theta',\varphi')}{\partial r'}\Big|_{r'=a} = \frac{r^2-a^2}{a(r^2+a^2-2ar\cos\gamma)^{3/2}}$$

This gives

$$\boxed{\phi(r,\theta,\varphi) = \frac{a(a^2-r^2)}{4\pi}\int\frac{d\Omega V(\Omega)}{(a^2+r^2-2ar\cos\gamma)^{3/2}}}$$

b) We may also expans the potential as

$$\boxed{\phi = \sum_{l=0}^{\infty}\sum_{m=-l}^{+l} A_{lm}Y_{lm}(\theta,\varphi)\left(\frac{r}{a}\right)^l}$$

To find the coefficients, it suffices to integrate the potential at $r=a$ against Y^*_{lm}:

$$\boxed{A_{lm} = \int_{\mathbb{S}^2} d\Omega V(\Omega)Y^*_{lm}(\Omega)}$$

This completes what we wanted to prove.

5 Jackson; 3.7

a) The potential is the sum of Coulomb terms

$$\boxed{\phi(r,\theta) = \frac{q}{4\pi\varepsilon_0 r}\left[\frac{1}{\sqrt{1+(a/r)^2-2(a/r)\cos\theta}}+\frac{1}{\sqrt{1+(a/r)^2+2(a/r)\cos\theta}}-2\right]}$$

In the quadrupole limit, this is

$$\boxed{\phi(r,\theta) = \frac{Q}{2\pi\varepsilon_0 r^3}\left(\frac{3\cos^2\theta-1}{2}\right)}$$

b) Using the method of images, the potential is the sum of 6 Coulomb terms

$$\begin{aligned}\phi(r,\theta) =& \frac{q}{4\pi\varepsilon_0 r}\Big\{\left[1+(a/r)^2-2(a/r)\cos\theta\right]^{-1/2}+\left[1+(a/r)^2+2(a/r)\cos\theta\right]^{-1/2}-2\\ &-b/a\left[1+(b^2/ar)^2-2(b^2/ar)\cos\theta\right]^{-1/2}-b/a\left[1+(b^2/ar)^2+2(b^2/ar)\cos\theta\right]^{-1/2}+\frac{2r}{b}\Big\}\end{aligned}$$

In the quadrupole limit, this becomes

$$\phi(r,\theta) = \frac{Q}{4\pi\varepsilon_0 r^3}\left\{3\cos^2\theta-1-\left(\frac{r^5}{b^5}\right)(3\cos^2\theta-1)\right\} = \boxed{\frac{Q}{2\varepsilon_0 r^3}P_2(\cos\theta)\left(1-\frac{r^5}{b^5}\right)}$$

6 Jackson; 3.11

In this problem, without loss of generality, we will work in the units system where $a = 1$. The factors of a may be properly re-introduced using dimensional analysis. We will also use x instead of ρ as the independent variable.

a) Using the Bessel equation, we may write

$$0 = \int_0^1 dx\ xJ_\alpha(qx)\Big[\frac{1}{x}\frac{d}{dx}x\frac{d}{dx}J_\alpha(kx) + \big(k^2 - \frac{\alpha^2}{x^2}\big)J_\alpha(kx)\Big]$$

$$= xJ_\alpha(qx)J'_\alpha(kx)\Big|_0^1 - \int_0^1 dx\ x\Big[J'_\alpha(kx)J'_\alpha(qx) + \big(k^2 - \frac{\alpha^2}{x^2}\big)J_\alpha(kx)J_\alpha(qx)\Big]$$

$$= -\lambda J_\alpha(k)J_\alpha(q) - \int_0^1 dx\ x\Big[J'_\alpha(kx)J'_\alpha(qx) + \big(k^2 - \frac{\alpha^2}{x^2}\big)J_\alpha(kx)J_\alpha(qx)\Big]$$

Antisymmetrizing both sides w.r.t k and q, we get

$$\boxed{(k^2 - q^2)\int_0^1 dx\ xJ_\alpha(kx)J_\alpha(qx) = 0}$$

b) First, let us show that these functions are indeed complete. The Bessel equation may be re-written in the eigen-function form

$$\Big[\frac{1}{x}\frac{d}{dx}x\frac{d}{dx} - \frac{\alpha^2}{x^2}\Big]\psi_n(x) = -y^2_{\alpha n}(\lambda)\psi_n(x)$$

all we need to show (at this level of mathematical rigor) is that the operator on the left hand side is Hermitean w.r.t the dot product

$$\langle\phi, \psi\rangle \equiv \int_0^1 dx\ x\phi^*(x)\psi(x)$$

over the set of functions with proper boundary conditions. The second term is clearly Hermitean, so let us deal with the first term.

$$\langle\phi, \frac{1}{x}\frac{d}{dx}x\frac{d}{dx}\psi\rangle = \int_0^1 dx\ \phi^*\frac{d}{dx}x\frac{d\psi}{dx}$$

$$= x\phi^*\psi'\Big|_0^1 - \int_0^1 dx\ \frac{d\phi^*}{dx}x\frac{d\psi}{dx}$$

$$= -\lambda\phi^*(1)\psi(1) - x\frac{d\phi^*}{dx}\psi\Big|_0^1 + \int_0^1 dx\ x\Big(\frac{1}{x}\frac{d}{dx}x\frac{d}{dx}\phi(x)\Big)^*\psi(x) = \Big\langle\frac{1}{x}\frac{d}{dx}x\frac{d}{dx}\phi, \psi\Big\rangle\ \blacksquare$$

So far, we know how to expand a function with the right boundary conditions as

$$|f\rangle = \sum_n A_n|n\rangle;\ \ A_n = \frac{\langle n|f\rangle}{\langle n|n\rangle}$$

It remains to compute the denominator. Once again, we use the Bessel equation to write

$$0 = \int_0^1 dx\ x^2J'_\alpha(kx)\Big[\frac{1}{x}\frac{d}{dx}x\frac{d}{dx}J_\alpha(kx) + \big(k^2 - \frac{\alpha^2}{x^2}\big)J_\alpha(kx)\Big]$$

$$= \frac{k}{2}J'^2(k) - \frac{\alpha^2}{2k}J^2(k) + k\int_0^1 dx\,x^2J_\alpha(kx)\frac{d}{dx}J_\alpha(kx)$$

$$= \frac{k}{2}J'^2(k) - \frac{\alpha^2}{2k}J^2(k) + kJ^2_\alpha(k) - 2k\langle k|k\rangle - k\int_0^1 dx\,x^2J_\alpha(kx)\frac{d}{dx}J_\alpha(kx)$$

$$= \frac{k}{2}J'^2(k) - \frac{\alpha^2}{2k}J^2(k) + \frac{k}{2}J^2_\alpha(k) - k\langle k|k\rangle$$

Where use has been made of $\alpha > 0$. Also, to get to the last line, we have substituted the whole equation with the arithmetic mean of the two above lines. Finally, we get

$$\boxed{\langle n|n\rangle = \frac{1}{2}\Big[J_\alpha'^2(k_n) + \big(1 - \alpha^2/k_n^2\big)J_\alpha^2(k_n)\Big]}$$