Electrodynamics III - HW#4

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1 Coulomb's Law From Fourier Transforms

In the Fourier space, Poisson's equation $\nabla^2\phi = -\rho/\varepsilon_0$ is written as

$$\tilde{\phi}(\mathbf{k}) = \frac{\tilde{\rho}(\mathbf{k})}{k^2\varepsilon_0}$$

1.1 $n \geq 3$ dimensions

For a point charge at $\mathbf{x}'$ and in $n \geq 3$ dimensions, we have

$$\tilde{\phi} = \frac{(2\pi)^{-n/2}}{k^2\varepsilon_0} e^{-i\mathbf{k}.\mathbf{x}'}$$

Which leads to the Green function

$$\phi = \frac{(2\pi)^{-n}}{\varepsilon_0}\int d^n\mathbf{k}\frac{e^{i\mathbf{k}.(\mathbf{x}-\mathbf{x}')}}{k^2} = \frac{\Omega_{n-2}}{(2\pi)^n\varepsilon_0}\int_0^\infty \frac{dk}{k}\int_0^\pi d\theta(k\sin\theta)^{n-2}e^{ikr\cos\theta}$$

$$= \frac{\Omega_{n-2}}{(2\pi)^n\varepsilon_0}\int_0^\infty dk\ k^{n-3}\int_{-1}^{+1} dx\ \cos(krx)(1-x^2)^{(n-3)/2}$$

$$= \frac{\Omega_{n-2}}{(2\pi)^n\varepsilon_0 r^{n-2}}\int_0^\infty dy\, y^{n-3}\int_{-1}^{+1} dx\ \cos(xy)(1-x^2)^{(n-3)/2}$$

Which is clearly proportional to r^{2-n}. From Gauss' law, we already know that this will be

$$\boxed{\phi = \frac{1}{r^{n-2}(n-2)\Omega_{n-1}\varepsilon_0}}$$

This is consistent with what we found only if

$$\int_0^\infty dy\, y^{n-3}\int_{-1}^{+1} dx\ \cos(xy)(1-x^2)^{(n-3)/2} = \frac{(2\pi)^n}{(n-2)\Omega_{n-1}\Omega_{n-2}}$$

using $\Omega_{n-1} = 2\frac{\pi^{n/2}}{\Gamma(n/2)}$ this is

$$\int_0^\infty dy\, y^{n-3}\int_{-1}^{+1} dx\ \cos(xy)(1-x^2)^{(n-3)/2} = \frac{\sqrt{\pi}2^n\Gamma\left(n/2\right)\Gamma\left(\frac{n-1}{2}\right)}{4(n-2)}$$

Define

$$\psi_n(y) \equiv \int_{-1}^{+1} dx\ \cos(xy)(1-x^2)^{(n-3)/2}$$

It is readily seen that these satisfy

$$\psi_n'' + \psi_n = \psi_{n+2}$$

Also, integration by parts allows us to write the first order derivative as

$$\psi_n' = -\int_{-1}^{+1} dx\, \sin(xy)\, x(1-x^2)^{(n-3)/2} = \frac{1}{y}\big[(n-3)\psi_{n-2} - (n-2)\psi_n\big]$$

Finally, we may also use integration by parts to get

$$\psi_{n+2} = \frac{n-1}{y^2}\big[(n-2)\psi_n - (n-3)\psi_{n-2}\big]$$

The three equations combine to give us

$$\psi_n'' + \frac{n-1}{y}\psi_n' + \psi_n = 0$$

In fact, there was a smart way to guess this directly, by interpreting ψ_n as a spherical wave form in n dimensions. In any case, $\psi_n(0) < \infty$ gives

$$\psi_n(x) = A_n \frac{J_{n/2-1}(x)}{y^{n/2-1}}$$

Comparison at $y = 0$ gives

$$A_n = \Gamma\big(\frac{n}{2}\big)2^{n/2} \int_0^1 dx\, (1-x^2)^{(n-3)/2} = \sqrt{\pi}2^{n/2-1}\Gamma\big(\frac{n-1}{2}\big)$$

At last, to prove consistency with the constant found by Gauss' law, we need to show

$$\int_0^\infty dy\, \big(\frac{y}{2}\big)^{(n-4)/2} J_{\frac{n}{2}-1}(y) = \Gamma\big(\frac{n}{2} - 1\big)$$

For $n = 3, 4$, this checks correct; beyond that, the RHS gives only the regularized value of the integral.

1.2 $n = 2$ dimensions

For 2D, the same approach gives

$$\phi = \frac{1}{4\pi^2\varepsilon_0} \int d^2\mathbf{k} \frac{e^{i\mathbf{k}.\mathbf{r}}}{k^2} = \frac{1}{4\pi^2\varepsilon_0} \int_0^\infty \frac{dk}{k} \int_0^{2\pi} e^{ikr\cos\varphi} d\varphi$$

Using Bessel's integral form, this is

$$\phi = \frac{1}{2\pi\varepsilon_0} \int_0^\infty \frac{dk}{k} J_0(kr)$$

To evaluate the integral, we use Frullani's identity

$$\int_0^\infty \frac{dx}{x}\big[f(ax) - f(bx)\big] = \log\big(\frac{a}{b}\big)\big[f(\infty) - f(0)\big]$$

In this case, this yields

$$\boxed{\phi(r) - \phi(r_0) = \frac{-1}{2\pi\varepsilon_0} \log\left(\frac{r}{r_0}\right)}$$

1.3 $n = 1$ dimensions

In 1D, the Fourier method breaks down and needs regularization techniques. The integral form is

$$\phi = \frac{1}{2\pi\varepsilon_0}\int \frac{dk}{k^2}e^{ikr}$$

Formally, this satisfies the Poisson equation

$$\phi'' = -\frac{1}{\varepsilon_0}\delta(x) \;\Rightarrow\; \phi = A + Bx - \frac{|x|}{2\varepsilon_0}$$

We pick the symmetric solution and drop the constant to get

$$\boxed{\phi = -\frac{|x|}{2\varepsilon_0}}$$

2 Q-Q interaction

In n dimensions, the quadrupole tensor is defined as

$$Q_{ij} = n\langle x_i x_j\rangle - \delta_{ij}\langle x^2\rangle; \quad \langle f(\mathbf{x})\rangle \equiv \int d^n\mathbf{x}\rho(\mathbf{x})f(\mathbf{x})$$

The interaction energy is a scalar, with proper dimensions, made out of two symmetric, traceless tensors and one vector. The most general form is

$$U = \frac{Q_{ij}Q'_{kl}}{(n-2)\Omega_{n-1}\varepsilon_0 r^{n+6}}\left[ar^4\delta_{ik}\delta_{jl} + br^2 r_j r_l\delta_{ik} + cr_i r_j r_k r_l\right]$$

where a, b, c are dimensionless constants to be found.

For a sample quadrupole consisting of 2 positive ($+q$) charges at $\mathbf{x}\pm a\hat{\mathbf{n}}$ and a $-2q$ charge located at $\mathbf{x}$, ($qa^2 = Q$ and a is very small) the quadrupole tensor is

$$Q_{ij} = Q(n\hat{\mathbf{n}}_i\hat{\mathbf{n}}_j - \delta_{ij})$$

and its energy is given by

$$U = Q\frac{\partial^2\phi}{\partial n^2}$$

Since all quadrupoles are a linear combination of such "rank-1" (except the δ_{ij} term that reduces the trace) quadrupoles, it costs us no generality to consider the example of a quadrupole at the origin, aligned with the z axis. The potential is given by

$$\phi = \frac{q}{(n-2)\Omega_{n-1}\varepsilon_0 r^{n+4}}\left[\left(1+\frac{a^2}{r^2}-2\frac{a}{r}\cos\theta\right)^{1-n/2} + \left(1+\frac{a^2}{r^2}+2\frac{a}{r}\cos\theta\right)^{1-n/2} - 2\right] = \frac{Q}{\Omega_{n-1}\varepsilon_0 r^n}(n\cos^2\theta - 1)$$

Then let Q'_{ij} be also of the same kind (rank-1) and located at a distance r, and an angle θ. We will consider different alignment directions each time:

i) If the second quadrupole is aligned along the radial direction, then the energy is

$$U = Q'\frac{\partial^2}{\partial r^2}\left[\frac{Q}{\Omega_{n-1}\varepsilon_0 r^n}(n\cos^2\theta - 1)\right] = \frac{QQ'(n\hat{\mathbf{z}}_i\hat{\mathbf{z}}_j - \delta_{ij})(n\hat{\mathbf{r}}_k\hat{\mathbf{r}}_l - \delta_{kl})}{(n-2)\Omega_{n-1}\varepsilon_0 r^{n+6}}\left[ar^4\delta_{ik}\delta_{jl} + br^2 r_j r_l\delta_{ik} + cr_i r_j r_k r_l\right]$$

this yields

$$na + (n-1)(b+c) = n(n+1)(n-2)$$

ii) If the second quadrupole is aligned along $\hat{\theta}$, then

$$U = Q'\big(\frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2\big)\big[\frac{Q}{\Omega_{n-1}\varepsilon_0 r^n}(n\cos^2\theta - 1)\big] = \frac{QQ'(n\hat{\mathbf{z}}_i\hat{\mathbf{z}}_j - \delta_{ij})(n\hat{\boldsymbol{\theta}}_k\hat{\boldsymbol{\theta}}_l - \delta_{kl})}{(n-2)\Omega_{n-1}\varepsilon_0 r^{n+6}}\big[ar^4\delta_{ik}\delta_{jl} + br^2 r_j r_l\delta_{ik} + cr_i r_j r_k r_l\big]$$

this time we get two equations

$$\begin{cases} 3n(n-2) = an(n-1) + b + c \\ (n-2)(n+4) = an + b + c \end{cases}$$

The bad news is that these new equations are not independent from the previous one and we will have to consider another case. The good news is that they are at least consistent and yield

$$\boxed{a = \frac{2(n-2)}{n}}, \quad b + c = n^2 - 4$$

iii) To find $b - c$, we need the second quadrupole to be neither perpendicular nor parallel to $\hat{\mathbf{r}}$. We also need to avoid the equator plane. So let us put the second quadrupole at $\theta = 45°$ and make it aligned with $\hat{\mathbf{s}}$. The energy is

$$U = Q'\big(\frac{1}{2r}\partial_r + \frac{1}{2}\partial_r^2 - \frac{1}{r^2}\partial_\theta + \frac{1}{2r^2}\partial_\theta^2\big)\big[\frac{Q}{\Omega_{n-1}\varepsilon_0 r^n}(n\cos^2\theta - 1)\big]\Big|_{\theta=\frac{\pi}{4}}$$

$$= \frac{QQ'(n\hat{\mathbf{z}}_i\hat{\mathbf{z}}_j - \delta_{ij})(n\hat{\mathbf{s}}_k\hat{\mathbf{s}}_l - \delta_{kl})}{(n-2)\Omega_{n-1}\varepsilon_0 r^{n+6}}\big[ar^4\delta_{ik}\delta_{jl} + br^2 r_j r_l\delta_{ik} + cr_i r_j r_k r_l\big]\Big|_{\theta=\frac{\pi}{4}}$$

After a relatively long calculation, this will yield

$$\boxed{c = \frac{n-2}{n}(n^2 + 2n + 8)}$$

and at last

$$\boxed{b = -\frac{8(n-2)}{n}}$$

In its full glory, the interaction formula is

$$\boxed{U = \frac{Q_{ij}Q'_{kl}}{n\Omega_{n-1}\varepsilon_0 r^{n+6}}\big[2r^4\delta_{ik}\delta_{jl} - 8r^2 r_j r_l\delta_{ik} + (n^2 + 2n + 8)r_i r_j r_k r_l\big]}$$