Electrodynamics III - HW#5

Koorosh Sadri - 400203113

December 2021

1 Classius-Mossotti Equation

In this model, each atom/molecule, is considered to be a sphere with the radius determined via

$$\frac{\delta N}{\delta V} = n = \frac{3}{4\pi R^3}$$

The particle (atom or molecule) *feels* some field $\mathbf{E}_{\text{else}}$ due to the presence of other particles and the external field and becomes polarized some amount

$$\mathbf{p} = \alpha \mathbf{E}_{\text{else}}$$

$\mathbf{E}_{\text{else}}$ at the origin is the same as the average field over the sphere since the Laplacian vanishes for $\mathbf{E}_{\text{else}}$. Therefore we have

$$\mathbf{E}_{\text{macroscopic}} = \mathbf{E}_{\text{else}} + \frac{3}{4\pi R^3}\int_{r\leq R} d^3\mathbf{x}\, \mathbf{E}_{\text{self}}$$

Where $\mathbf{E}_{\text{self}}$ is the field due to the dipole $\mathbf{p}$ at the origin. According to Jackson's eq. 4.18, and the linear relation ship $\mathbf{p} = \alpha \mathbf{E}_{\text{else}}$ we get

$$\mathbf{E}_{\text{macroscopic}} = \mathbf{E}_{\text{else}} - \frac{\mathbf{p}}{4\pi\varepsilon_0 R^3} = \Big(1 - \frac{\alpha}{4\pi\varepsilon_0 R^3}\Big)\mathbf{E}_{\text{else}}$$

Finally

$$\mathbf{P} = \chi_e \varepsilon_0 \mathbf{E}_{\text{macroscopic}} = \frac{3\mathbf{p}}{4\pi R^3} = \frac{3\alpha}{4\pi R^3}\Big(1 - \frac{\alpha/\varepsilon_0}{4\pi R^3}\Big)^{-1}\mathbf{E}_{\text{macroscopic}}$$

which yields

$$\boxed{\chi_e \equiv \kappa_e - 1 = \frac{n\alpha/\varepsilon_0}{1 - \frac{n\alpha}{3\varepsilon_0}}}$$

2 Jackson; 3.15

a) Inside and outside the sphere, $\boldsymbol{\nabla}.\mathbf{J} = 0$ implies the Laplace equation for the electrostatic potential. On the boundary, the potential is continuous while the normal component of the electric field satisfies

$$\sigma' E_r^+ = \sigma E_r^- + \sigma F \cos\theta$$

This leads to the unique potential

$$\phi = \frac{aF\sigma}{\sigma + 2\sigma'}\cos(\theta)\begin{cases} \frac{r}{a} & r \leq R \\ \frac{a^2}{r^2} & r \geq R \end{cases}$$

comparison with the potential form for a dipole we get

$$\boxed{\mathbf{p} = \frac{4\pi\varepsilon_0 a^3 F\sigma}{\sigma + 2\sigma'}\hat{\mathbf{z}}}$$

b)

$$I = \sigma' \int_0^{\frac{\pi}{2}} 2\pi a^2 \sin\theta \, d\theta \, \frac{2F\sigma}{\sigma + 2\sigma'} \cos\theta = \boxed{\frac{2\pi F\sigma\sigma' a^2}{\sigma + 2\sigma'}}$$

$$P_e = 2\pi\sigma' \int_a^\infty dr\, r^2 \int_0^\pi d\theta \sin\theta \, \frac{F^2\sigma^2 a^6}{(\sigma+2\sigma')^2} \frac{1+3\cos^2\theta}{r^6} = \frac{8\pi F^2\sigma^2\sigma' a^3}{3(\sigma+2\sigma')^2}$$

therefore

$$\boxed{R_e = \frac{P_e}{I^2} = \frac{2}{3\pi a\sigma'}}; \quad \boxed{V_e = IR_e = \frac{4Fa\sigma}{3(\sigma+2\sigma')}}$$

c)

$$P_i = \frac{4}{3}\pi a^3 \sigma \Big(F - \frac{F\sigma}{\sigma+2\sigma'}\Big)^2 = \boxed{\frac{16\pi a^3 F^2 \sigma\sigma'^2}{3(\sigma+2\sigma')^2}}$$

therefore

$$\boxed{r = \frac{P_i}{I^2} = \frac{4}{3\pi a\sigma}}$$

d)

$$\boxed{V_t = I(R_e + r) = \frac{4}{3}Fa}$$

$$IV_t = \frac{4}{3}\pi a^3 \sigma F \frac{2F\sigma'}{\sigma+2\sigma'} = \frac{4}{3}\pi a^3 \sigma F E_i = P_{Bat.}$$

3 Jackson; 4.1

a) For the configuration (a):

$$q_{lm} = qa^l\Big[Y^*_{lm}(\frac{\pi}{2},0) + Y^*_{lm}(\frac{\pi}{2},\frac{\pi}{2}) - Y^*_{lm}(\frac{\pi}{2},\pi) - Y^*_{lm}(\frac{\pi}{2},-\frac{\pi}{2})\Big]$$

$$= qa^l\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(0)[1+(-i)^m-(-1)^m-i^m]$$

For even m, this is zero, for $m = 2r+1$, this is

$$q_{l,2r+1} = 2qa^l\sqrt{\frac{2l+1}{4\pi}\frac{(l-2r-1)!}{(l+2r+1)!}}P_l^{2r+1}(0)\big[1-(-)^r i\big]$$

For even l, this also vanishes, for $l = 2s+1$, we may use Jackson's eq. 3.50 to find

$$q_{2s+1,2r+1} = qa^{2s+1}\sqrt{\frac{4s+3}{4\pi}\frac{(2s-2r)!}{(2s+2r+2)!}}\frac{(-)^{s+r+1}}{4^s}\frac{(2s+2r+2)!}{(s-r)!(s+r+1)!}\big[1-(-)^r i\big]$$

which further simplifies to

$$\boxed{q_{2s+1,2l+1} = q\frac{a^{2s+1}}{(-4)^s}\sqrt{\frac{4s+3}{4\pi}\binom{2s-2r}{s-r}\binom{2s+2r+2}{s+r+1}}\big[(-1)^{r+1}+i\big]}$$

The first few moments are

$$\boxed{q_{1,\pm 1} = qa\sqrt{\frac{3}{2\pi}}(\mp 1 + i)}$$

$$\boxed{q_{3,\pm 1} = \frac{qa^3}{4}\sqrt{\frac{21}{\pi}}(\pm 1 - i)}$$

$$\boxed{q_{3,\pm 1} = \frac{qa^3}{4}\sqrt{\frac{35}{\pi}}(\pm 1 - i)}$$

For the configuration (b) only even l ($l = 2s > 0$) and zero m survive

$$\boxed{q_{2s,0} = qa^{2s}\sqrt{\frac{4s+1}{\pi}}; \quad s = 1, 2, \cdots}$$

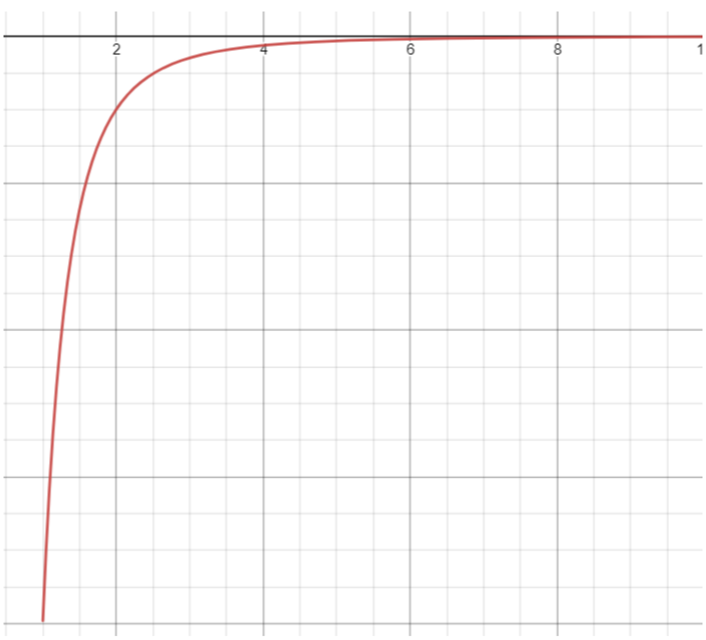
c) The potential is expanded as

$$\boxed{\phi = \frac{q}{2\pi\varepsilon_0 r}\sum_{s=1}^{\infty}\left(\frac{a}{r}\right)^{2s} P_{2s}(\cos\theta)}$$

on the equator plane, the dominant term is

$$\phi = -\frac{qa^2}{4\pi\varepsilon_0 r^3}$$

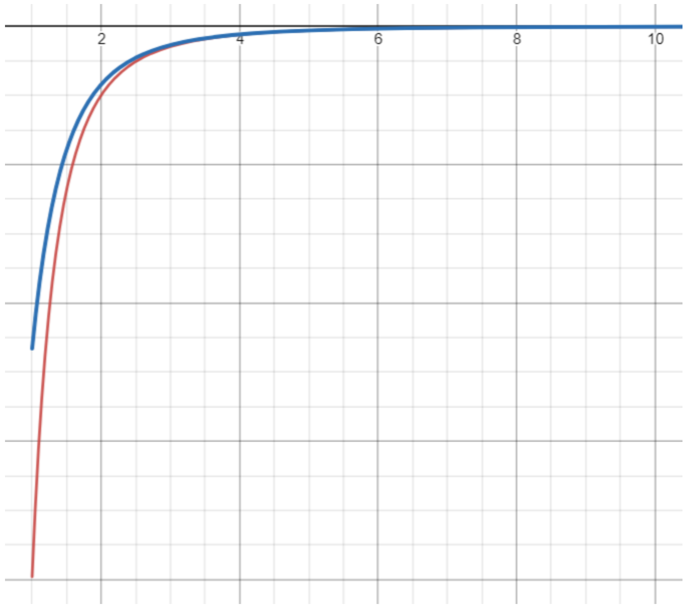
In the units where $q = a = \varepsilon_0 = 1$, this looks like below



d) From Coulomb's law, the potential is

$$\phi = \frac{q}{2\pi\varepsilon_0}\left(\frac{1}{\sqrt{r^2 + a^2}} - \frac{1}{r}\right)$$

which is plotted as



4 Jackson; 4.7

a,b) Instead of using the q_{lm} moments, we expand the potential as

$$\phi = \sum_{l=0}^{\infty} \psi_l(r) P_l(\cos\theta)$$

the charge density is written as

$$\rho = \frac{2}{3}\rho_0\big(\frac{r}{a}\big)^2 e^{-r/a}\big[P_0(\cos\theta) - P_2(\cos\theta)\big]$$

The Poisson equation then becomes

$$\big[\frac{d}{dr}r^2\frac{d}{dr} - l(l+1)\big]\psi_l(r) = -\frac{r^2}{\varepsilon_0}\begin{cases} \frac{2}{3}\rho_0\big(\frac{r}{a}\big)^2 e^{-r/a} & l=0 \\ -\frac{2}{3}\rho_0\big(\frac{r}{a}\big)^2 e^{-r/a} & l=2 \\ 0 & o.w. \end{cases}$$

From now on, we will use the system of units in which $\rho_0 = a = \varepsilon_0 = 1$. (The answers will be $64\pi\varepsilon_0$ times Jackson's). For $l = 0$, the equation integrates as

$$\psi_0'(r) = \frac{2}{3}e^{-r}\big(r^2 + 4r + 12 + \frac{24}{r} + \frac{24}{r^2} + \frac{Ae^r}{r^2}\big)$$

The choice $A = -24$ is the only one that guarantees finite field at the origin. Then

$$\psi_0(r) = B - \frac{2}{3}\frac{r^3 + 6r^2 + 18r + 24(1-e^r)}{re^r}$$

To set $\phi(\infty) = 0$, we find $B = 0$ and

$$\boxed{\psi_0(r) = -\frac{2}{3}\frac{r^3 + 6r^2 + 18r + 24(1-e^r)}{re^r}}$$

For ψ_2 we have

$$\big(\frac{d}{dr}r^2\frac{d}{dr} - 6\big)\psi_2(r) = \frac{2}{3}r^4 e^{-r}$$

A series expansion gives

$$\psi_2(r) = Cr^2 + \frac{2}{3}\sum_{n=4}^{\infty} \frac{(-)^n r^n}{(n-4)!(n-2)(n+3)}$$

$$= Cr^2 + \frac{2}{3}\frac{r^5e^r + 720(1 + r + r^2/2 + r^3/6 - e^r) + 30r^4 + 5r^5}{5r^3e^r}$$

Which clearly gives $C = -2/15$

$$\boxed{\psi_2(r) = \frac{288(1 + r + r^2/2 + r^3/6 - e^r) + 12r^4 + 2r^5}{3r^3e^r}}$$

Therefore the potential is

$$\phi = \psi_0(r) + \psi_2(r)\frac{3\cos^2\theta - 1}{2}$$

In the large r limit, we may drop the exponentially decaying terms and get

$$\psi_0(r) \approx \frac{16}{r}; \quad \psi_2(r) \approx -\frac{96}{r^3}$$

And therefore

$$\boxed{\phi \approx \frac{16}{r}\left[1 - \frac{3}{r^2}(3\cos^2\theta - 1)\right]}$$

Finally, near the origin

$$\boxed{\phi \approx 4 - \frac{2}{15}r^2 P_2(\cos\theta)}$$

c) Bringing back the unit constants, we know that for large r, the monopole term is

$$\phi = \frac{16\rho_0 a^3}{\varepsilon_0 r} = \frac{e}{4\pi\varepsilon_0 r}$$

therefore

$$e = 64\pi\rho_0 a^3$$

The interaction term is given by

$$U = ZeQ\partial_z^2\phi$$

at the origin, this is

$$U = ZeQ \times \frac{4\rho_0}{15\varepsilon_0}$$

The total sign in the equation above is irrelevant. Substituting ρ_0 we get

$$\nu = \frac{U}{h} = \frac{Ze^2Q}{80\pi\varepsilon_0 a^3 h} = \frac{ZQ\alpha c}{120\pi a^3}$$

For a hydrogen atom ($Z = 1$), this is

$$\boxed{\nu \approx 3.9\,\mathrm{MHz}}$$

5 Jackson; 4.13

Far away from the edge where the liquid ends, the potential is given as

$$\phi = V\frac{\log(s/a)}{\log(b/a)}$$

Therefore the free charge per unit length is

$$cV = \frac{2\pi\varepsilon V}{\log(b/a)}$$

therefore the capacitance per unit length is

$$c = \frac{2\pi\varepsilon}{\log(b/a)}$$

If the liquid rises an infinitesimal amount dh, then the energy *increases* as

$$dE = \frac{\pi\varepsilon_0\chi_e V^2}{\log(b/a)}dh$$

meanwhile, the voltage supplies need to do work in order to keep the potential difference unchanged. The work done is

$$dW = V\,dQ = V^2 dc = \frac{2\pi\varepsilon_0\chi_e V^2}{\log(b/a)}dh$$

therefore, the difference is due to the potential difference and we get

$$U = -\frac{\pi\varepsilon_0\chi_e V^2}{\log(b/a)}h$$

Adding the gravitational energy, this is

$$U = -\frac{\pi\varepsilon_0\chi_e V^2}{\log(b/a)}h + \frac{\pi}{2}\rho g(b^2 - a^2)h^2$$

The equilibrium is at

$$h = \frac{\chi_e\varepsilon_0 V^2}{\rho g(b^2 - a^2)\log(b/a)}$$

which gives

$$\boxed{\chi_e = \frac{\rho g h(b^2 - a^2)\log(b/a)}{\varepsilon_0 V^2}}$$