Electrodynamics III - HW#5

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1 Jackson; 5.1

For any constant vector $\mathbf{n}$, we have

$$\frac{4\pi}{\mu_0 I}\mathbf{B}.\mathbf{n} = \oint \Big(d\mathbf{x}' \times \boldsymbol{\nabla}' \frac{1}{|\mathbf{x}-\mathbf{x}'|}\Big).\mathbf{n}$$

$$= \oint d\mathbf{x}'.\Big[\boldsymbol{\nabla}'(\frac{1}{|\mathbf{x}-\mathbf{x}'|}) \times \mathbf{n}\Big]$$

$$= \int d\mathbf{a}'.\boldsymbol{\nabla}' \times \Big[\boldsymbol{\nabla}'(\frac{1}{|\mathbf{x}-\mathbf{x}'|}) \times \mathbf{n}\Big]$$

$$= \int d\mathbf{a}'.\Big[\mathbf{n}.\boldsymbol{\nabla}'\boldsymbol{\nabla}'(\frac{1}{|\mathbf{x}-\mathbf{x}'|})\Big]$$

$$= -\mathbf{n}.\boldsymbol{\nabla} \int d\mathbf{a}'.\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^2} = \mathbf{n}.\boldsymbol{\nabla}\Omega$$

Which is equivalent to

$$\boxed{\mathbf{B} = \frac{\mu_0 I}{4\pi}\boldsymbol{\nabla}\Omega}$$

2 Jackson; 5.17

a, b) All we need to show is that the suggested fields satisfy Maxwell's equations

$$\boldsymbol{\nabla} \times \mathbf{H} = \mathbf{J}_{free}$$

with proper boundary conditions

$$\Delta(\hat{\mathbf{z}} \times \mathbf{H}) = 0; \quad \Delta(\hat{\mathbf{z}}.\frac{\mathbf{B}}{\mu}) = 0$$

Maxwell's equation is clearly satisfied since in each region, the fields correspond to the free sources and the image sources that exist in the other region. To see how the tangent boundary condition is met, consider the x component of H and write the contribution from the image currents as a modified integral over the $z > 0$ region

$$H_x^{z>0} = \int_{z>0} \frac{d^3\mathbf{x}'}{4\pi|\mathbf{x}-\mathbf{x}'|^3}\Big[J_y(\mathbf{x}')(-z') - J_z(\mathbf{x}')(y-y')\Big] + \int_{z>0} \frac{d^3\mathbf{x}'}{4\pi|\mathbf{x}-\mathbf{x}'|^3}\Big[\frac{\mu_r-1}{\mu_r+1}J_y(\mathbf{x}')(+z') + \frac{\mu_r-1}{\mu_r+1}J_z(\mathbf{x}')(y-y')\Big]$$

$$= \frac{2}{\mu_r+1}\int_{z>0} \frac{d^3\mathbf{x}'}{4\pi|\mathbf{x}-\mathbf{x}'|^3}\Big[J_y(\mathbf{x}')(-z') - J_z(\mathbf{x}')(y-y')\Big] = H_x^{z<0}$$

And for the normal boundary conditions

$$B_z^{z>0} = \mu_0 \int_{z>0} \frac{d^3\mathbf{x}'}{4\pi|\mathbf{x}-\mathbf{x}'|^3}\Big[J_x(\mathbf{x}')(y-y')-J_y(\mathbf{x}')(x-x')\Big]+\mu_0\frac{\mu_r-1}{\mu_r+1}\int_{z>0}\frac{d^3\mathbf{x}'}{4\pi|\mathbf{x}-\mathbf{x}'|^3}\Big[J_y(\mathbf{x}')(-z')-J_z(\mathbf{x}')(y-y')\Big]$$

$$= \mu_0\frac{2\mu_r}{\mu_r+1}\int_{z>0}\frac{d^3\mathbf{x}'}{4\pi|\mathbf{x}-\mathbf{x}'|^3}\Big[J_y(\mathbf{x}')(-z')-J_z(\mathbf{x}')(y-y')\Big] = B_z^{z<0}$$

This completes our proof. Note that we have proved this:

In $z>0$, solve the problem assuming there is no magnetic material ($\mu_r = 1$ everywhere) and take into account the image currents as given by Jackson. In $z<0$, pretend the world is uniformaly filled with the magnetic material and the free *current is the same as the source, attenuated by a factor $2/(\mu_r+1)$.*

3 Jackson; 5.25

a) If the wire is on the z-axis and the current is running in the plus direction, then

$$\mathbf{B}_2 = \frac{\mu_0 I_2}{2\pi}\frac{-y\hat{\mathbf{x}}+x\hat{\mathbf{y}}}{(x^2+y^2)}$$

Meanwhile, the surface element vector is

$$d\mathbf{a} = a\,d\ell(-\sin\alpha\hat{\mathbf{x}}+\cos\alpha\hat{\mathbf{y}})$$

Therefore

$$F_2 = \frac{\mu_0 I_2 a}{2\pi}\int_{-b/2}^{+b/2} d\ell\frac{\ell+d\cos\alpha}{d^2+\ell^2+2d\ell\cos\alpha} = \frac{\mu_0 I_2 a}{4\pi}\int_{-b/2}^{+b/2} d\ell\frac{2(\ell+d\cos\alpha)}{d^2\sin^2\alpha+(\ell+d\cos\alpha)^2}$$

$$= \frac{\mu_0 I_2 a}{4\pi}\ln\frac{d^2+b^2/4+bd\cos\alpha}{d^2+b^2/4-bd\cos\alpha}$$

Finally,

$$\boxed{W = I_1 F_2 = \frac{\mu_0 I_1 I_2 a}{4\pi}\ln\frac{d^2+b^2/4+bd\cos\alpha}{d^2+b^2/4-bd\cos\alpha}}$$

b) If we use current sources to keep the currents constant, then as the loop moves a distance δd away from the wire, each batter does some work $dW = \frac{\partial W}{\partial d}\delta d$. The kinetic energy however, changes only as $dT = F\delta d$ where F is the repulsive force. the difference is the increase in potential energy. That is

$$2\delta W = F\delta d + \delta U$$

but we know that the potential energy is the same as the interaction W, therefore (Cf. Jackson's eq. 5.151)

$$F = \frac{\partial W}{\partial d} = \frac{\mu_0 I_1 I_2 a}{4\pi}\Big(\frac{2d+b\cos\alpha}{d^2+b^2/4+bd\cos\alpha} - \frac{2d-b\cos\alpha}{d^2+b^2/4-bd\cos\alpha}\Big)$$

$$= \boxed{-\frac{\mu_0 I_1 I_2 ab\cos\alpha}{2\pi}\frac{d^2-b^2/4}{(d^2+b^2/4)^2-b^2d^2\cos^2\alpha}}$$

c) This time the surface element is

$$d\mathbf{a} = 2\sqrt{a^2-\ell^2}d\ell(-\sin\alpha\hat{\mathbf{x}}+\cos\alpha\hat{\mathbf{y}})$$

and therefore

$$F_2 = \frac{\mu_0 I_2}{\pi}\int_{-a}^{+a} d\ell\,\sqrt{a^2-\ell^2}\frac{\ell+d\cos\alpha}{d^2+\ell^2+2d\ell\cos\alpha}$$

$$= \frac{\mu_0 I_2 a}{2\pi} \int_{-1}^{+1} dx\, \sqrt{1-x^2}\, \frac{1}{x + (d/a)e^{i\alpha}} \;\; + c.c.$$

$$= \frac{\mu_0 I_2 a}{2\sqrt{\pi}} \,\mathrm{Re} \Big\{ \sum_{n=0}^{\infty} \big(\frac{a}{d}\big)^{2n+1} e^{i(2n+1)\alpha} \frac{\Gamma(n+1/2)}{\Gamma(n+2)} \Big\}$$

$$= \mu_0 I_2 a\, \mathrm{Re} \Big\{ \frac{1 - \sqrt{1-(ae^{i\alpha}/d)^2}}{(a/d)e^{i\alpha}} \Big\}$$

$$= \mu_0 I_2 d\, \mathrm{Re} \left[e^{-i\alpha} - \sqrt{e^{-2i\alpha} - a^2/d^2} \right]$$

which yields

$$\boxed{W = \mu_0 I_1 I_2 d\, \mathrm{Re} \left[e^{-i\alpha} - \sqrt{e^{-2i\alpha} - a^2/d^2} \right]}$$

$$= \mu_0 I_1 I_2 \Big[d\cos\alpha - \frac{1}{\sqrt{2}} \Big(d^2 \cos 2\alpha - a^2 + \sqrt{d^4 - 2a^2 d^2 \cos 2\alpha + a^4} \Big)^{1/2} \Big]$$

The repulsive force is

$$F = \frac{\partial W}{\partial d} = \boxed{\mu_0 I_1 I_2 \Big[\cos\alpha - \frac{1}{\sqrt{2}} \Big(d^2 \cos 2\alpha - a^2 + \sqrt{d^4 - 2a^2 d^2 \cos 2\alpha + a^4} \Big)^{-1/2} \Big(d \cos 2\alpha + \frac{d^3 - a^2 d \cos 2\alpha}{\sqrt{d^4 - 2a^2 d^2 \cos 2\alpha + a^4}} \Big) \Big]}$$

a careful evaluation shows that the sign is indeed negative and therefore it is an attractive force in fact. Just like the rectangular loop.

d) The Rectangular Loop:

$$W \approx \frac{\mu_0 I_1 I_2 a}{4\pi} \frac{2bd\cos\alpha}{d^2} = I_1 ab \cos\alpha \frac{\mu_0 I_2}{2\pi d}$$

which is the same as $\mathbf{m}.\mathbf{B}$.

The circular loop:

$$W \approx \mu_0 I_1 I_2 d\, \mathrm{Re} \left[e^{-i\alpha} (1 - 1 + a^2 e^{2i\alpha} 2d^2) \right] = \frac{\mu_0 I_2}{2\pi d} \pi a^2 \cos\alpha = \mathbf{m}.\mathbf{B}$$

The sign convention corresponds to the work function. This will lead to the force via the positive derivative. However, one might want to construct a potential from the force, then they would write $U = -\mathbf{m}.\mathbf{B}$.

4 Jackson; 5.26

Let us compute the energy per unit length as

$$\frac{1}{2} L I^2 = \frac{1}{2} \int \mathbf{A}.\mathbf{J}\, da$$

Therefore

$$L = \frac{1}{\pi a^2} \int_{\text{wire a}} da\, \frac{A_b + A_a}{I} \;\; + \;\; a \leftrightarrow b$$

Inside and outside a wire, the (per current) potential is

$$A^a_{ins} = C_a - \frac{\mu_0 s^2}{4\pi a^2}; \;\; A^a_{out} = -\frac{\mu_0}{2\pi} \ln \frac{s}{D_a}; \;\; \cdots$$

continuity gives

$$C_a = \frac{\mu_0}{4\pi}\big(1 - 2\log(a/D_a)\big)$$

Also, if we wish the potential to vanish at infinity, then $D_a = D_b = D$. This results in

$$L = \frac{\mu_0}{4\pi}\big[1 - 2\log\frac{a}{D}\big] - \frac{\mu_0}{8\pi} - \frac{\mu_0}{2\pi}\log D + \frac{\mu_0}{2\pi}\frac{1}{\pi a^2}\int_{\text{wire a}} da\ \log(s') \ + \ a \leftrightarrow b$$

The harmonicity of the log function in 2D, guarantees that

$$\frac{1}{\pi a^2}\int_{\text{wire a}} da\ \log(s') = \log(d)$$

Therefore

$$\begin{aligned} L &= \frac{\mu_0}{4\pi}\big[2 - 2\log ab\big] - \frac{\mu_0}{4\pi} + \frac{\mu_0}{\pi}\log d \\ &= \boxed{\frac{\mu_0}{4\pi}\big[1 - 2\log\frac{ab}{d^2}\big]} \end{aligned}$$

5 Jackson; 5.29

Let ψ be the unique scalar defined on the 2D cross section plane outside the wires satisfying

$$\nabla^2\psi = 0; \quad \oint_{\partial\, \pm\, \text{wire}} \boldsymbol{\nabla}\psi . d\boldsymbol{\ell}_\perp = \pm 1; \quad \psi = \pm\psi_0 \text{ on the } \pm \text{ wire}$$

where ψ_0 is a constant that is found after solving the boundary conditions problem. Then, to find the capacity, we put charger per unit length $\pm\lambda$ on the wires. Then, the electrostatic potential will be

$$\phi = \frac{\lambda}{\varepsilon}\psi$$

and therefore the voltages are

$$V = \pm\frac{Q}{2C} = \pm\frac{\lambda l}{2C} = \pm\frac{\lambda}{\varepsilon}\psi_0$$

Or

$$\boxed{C = \frac{\varepsilon l}{2\psi_0}}$$

To compute the self inductance, we first note that the perfect conductors do not allow any of the magnetic fields to exist inside them. The magnetic vector potential

$$\mathbf{A} = \mu I\psi\hat{\mathbf{z}}$$

solves the problem and we have

$$W = \frac{1}{2}LI^2 = \frac{1}{2}\int \mathbf{A}.\mathbf{J} = \frac{1}{2}\int_{\partial\text{wires}} dlKA = \mu\psi_0 I^2 l$$

or

$$\boxed{L = 2\mu\psi_0 l}$$

Finally

$$\boxed{\frac{L}{l}\frac{C}{l} = \varepsilon\mu}$$

6 Jackson; 5.35

a) Let us start with the vector potential

$$\mathbf{A} = \hat{\boldsymbol{\varphi}}\frac{B_0}{2}\sin\theta \begin{cases} r & r \leq a \\ \\ \frac{a^3}{r^2} & r \geq a \end{cases}$$

which yields the correct field as

$$\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A} = B_0 \begin{cases} \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}} & r < a \\ \\ \frac{a^3}{2r^3}\big(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}\big) & r > a \end{cases}$$

Finally, using $\Delta\mathbf{B} = -\mu\hat{\mathbf{r}}\times\mathbf{K}$, we find

$$\boxed{\mathbf{K} = \frac{3B_0}{2\mu}\hat{\boldsymbol{\varphi}}}$$

b) For an azimuthal field

$$\mathbf{A} = A(t,r,\theta)\hat{\boldsymbol{\varphi}}$$

we have

$$\nabla^2\mathbf{A} = -\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\mathbf{A} = \frac{\hat{\varphi}}{r^2}\big(r\partial_r^2 r + \partial_\theta\frac{1}{\sin\theta}\partial_\theta\sin\theta\big)A$$

To solve the differential equation $\nabla^2 = \mu\sigma$, we first switch to units where $\mu\sigma = B_0 = a = 1$. Then, the differential equation is

$$\frac{1}{r^2}\big(r\partial_r^2 r + \partial_\theta\frac{1}{\sin\theta}\partial_\theta\sin\theta\big)A = \partial_t A$$

Separating variables is helpful since the initial conditions are proportional to $\sin\theta$ which is an eigenvalue of the θ operator. Therefore, the solution is $A = a(r,t)\sin\theta$ with

$$\partial_t a = \frac{1}{r^2}\big(-2 + r\partial_r^2 r\big)a$$

Once again, we use separation of variables to get

$$a = \alpha(r)e^{-st};\quad \frac{d^2\alpha}{dr^2} + \frac{2}{r}\frac{d\alpha}{dr} + (s - \frac{2}{r^2})\alpha = 0$$

This is solved as

$$A(t,r,\theta) = \sin\theta\int ds\, e^{-st}\Big[C(s)j_1(\sqrt{s}r) + D(s)y_1(\sqrt{s}r)\Big]$$

Stability confines the integral to positive s, then the smooth initial conditions at $r = 0$ imply $D(s) = 0$. Therefore

$$A = \sin\theta\int_0^\infty ds\, C(s)e^{-st}j_1(\sqrt{s}r)$$

comparison with the initial conditions, determines the amplitudes $C(s)$. We just need to use the orthogonality relation

$$\int_0^\infty dx\, x^2\, j_1(ux)j_1(vx) = \frac{\pi}{2u^2}\delta(u - v)$$

to find

$$C(s) = \frac{\sqrt{s}}{2\pi}\Big[\int_0^1 dr\, r^3 j_1(\sqrt{s}r)\ +\ \int_1^\infty dr\, j_1(\sqrt{s}r)\Big]$$

$$= \frac{1}{2\pi}\Big[\frac{1}{s\sqrt{s}}\int_0^{\sqrt{s}} dx\,(x\sin x\ -\ x^2\cos x)\ +\ \int_{\sqrt{s}}^{\infty} dx\,\frac{\sin x\ -x\cos x}{x^2}\Big]$$

$$= \frac{1}{2\pi}\Big\{\frac{-1}{s\sqrt{s}}\Big[(s-3)\sin\!\big(\sqrt{s}\big)+3\sqrt{s}\cos\!\big(\sqrt{s}\big)\Big]+\frac{\sin(\sqrt{s})}{\sqrt{s}}\Big\}$$

$$= \frac{3}{2\pi\sqrt{s}}j_1(\sqrt{s})$$

Which means

$$\boxed{\mathbf{A} = \frac{3\hat{\boldsymbol{\varphi}}}{2\pi}\sin\theta\int_0^{\infty}\frac{ds}{\sqrt{s}}e^{-st}j_1(\sqrt{s})j_1(\sqrt{s}r)}$$

Near the origin, and up to the first order in r, we have

$$\mathbf{A} \approx (r\sin\theta\,\hat{\boldsymbol{\varphi}})\frac{1}{2\pi}\int_0^{\infty}\frac{ds}{s}e^{-st}(\sin\sqrt{s}\ -\ \sqrt{s}\cos\sqrt{s})$$

which gives

$$\mathbf{B}(\mathbf{0},t) = \frac{\hat{\mathbf{z}}}{\pi}\int_0^{\infty}\frac{ds}{s}e^{-st}(\sin\sqrt{s}\ -\ \sqrt{s}\cos\sqrt{s})$$

$$= \frac{-2\hat{\mathbf{z}}}{\sqrt{\pi t}}\sum_{n=0}^{\infty}\frac{n}{(2n+1)!}\Big(\frac{-1}{2t}\Big)^n\langle X^{2n}\rangle$$

for a standard Gaussian variable X. Using Wick's theorem, this is

$$\mathbf{B}(\mathbf{0},t) = \frac{-2\hat{\mathbf{z}}}{\sqrt{\pi t}}\sum_{n=0}^{\infty}\frac{n}{(2n+1)}\Big(\frac{-1}{4t}\Big)^n\frac{1}{n!}$$

$$= \frac{-\hat{\mathbf{z}}}{\sqrt{\pi t}}\sum_{n=0}^{\infty}(1-\frac{1}{2n+1})\Big(\frac{-1}{4t}\Big)^n\frac{1}{n!}$$

$$= \frac{-\hat{\mathbf{z}}}{\sqrt{\pi t}}e^{-1/4t}+\frac{1}{\sqrt{\pi t}}\sum_{n=0}^{\infty}\frac{1}{(2n+1)\times n!}\Big(\frac{-1}{4t}\Big)^n$$

$$= \boxed{\hat{\mathbf{z}}\Big[\mathrm{erf}\Big(\frac{1}{2\sqrt{t}}\Big)-\frac{e^{-1/4t}}{\sqrt{\pi t}}\Big]}$$

c)

$$W_m = 2\pi\int_0^{\infty}dr\,r^2\int_0^{\pi}d\theta\,\sin\theta\,\frac{B_r^2+B_\theta^2}{2}$$

$$= \pi\int_0^{\infty}dr\,r^2\int_0^{\pi}d\theta\sin\theta\,\Big(\frac{4a^2}{r^2}\cos^2\theta+\frac{a^2\sin^2\theta}{r^2}+a'^2\sin^2\theta+\frac{2aa'\sin^2\theta}{r}\Big)$$

$$= 4\pi\int_0^{\infty}dr\,\Big(a^2+\frac{1}{3}r^2a'^2+\frac{2raa'}{3}\Big)$$

$$= \frac{9}{\pi}\int_0^{\infty}ds\,e^{-st}j_1(\sqrt{s})\int_0^{\infty}ds'\,e^{-s't}j_1(\sqrt{s'})\int_0^{\infty}dr\Big[\frac{j_1(\sqrt{s}r)j_1(\sqrt{s'}r)}{\sqrt{ss'}}+\frac{r^2}{3}j_1'(\sqrt{s}r)j_1'(\sqrt{s'}r)+\frac{2r}{3\sqrt{s}}j_1(\sqrt{s}r)j_1'(\sqrt{s'}r)\Big]$$

$$= \frac{9}{\pi}\int_0^{\infty}ds\,e^{-st}j_1(\sqrt{s})\int_0^{\infty}ds'\,e^{-s't}j_1(\sqrt{s'})\int_0^{\infty}dr\Big[\frac{2j_1(\sqrt{s}r)j_1(\sqrt{s'}r)}{\sqrt{ss'}}+\frac{r^2}{3}j_2(\sqrt{s}r)j_2(\sqrt{s'}r)-\frac{4r}{3\sqrt{s}}j_1(\sqrt{s}r)j_2(\sqrt{s'}r)\Big]$$

Where in the last line, I have used the identity

$$j_l'(x) = \frac{l}{x}j_l(x)-j_{l+1}(x)$$

According to [1], (and [2]), we have the following integral

$$\int_0^\infty dx\, j_l(kx) j_l(k'x) \;=\; \frac{\pi}{2(2l+1)} \frac{k_<^l}{k_>^{l+1}}$$

In addition, we may write

$$\int_0^\infty dx\, x j_1(ux) j_2(vx) = \frac{1}{u^2} F(u/v)$$

Partial differentiation with respect to u yields

$$-\frac{2}{u^3}F(u/v) + \frac{1}{u^2 v}F'(u/v) = \int_0^\infty dx\, x^2 \Big[\frac{j_1(ux)}{ux} - j_2(ux)\Big] j_2(vx) \;=\; \frac{1}{u^3}F(u/v) - \frac{\pi}{2v^3}\delta(u/v-1)$$

This is equivalent to

$$F'(x) - \frac{3}{x}F(x) = -\frac{\pi}{2}\delta(x-1)$$

which is solved as

$$F(x) = x^3 \begin{cases} A & x<1 \\ \\ A - \frac{\pi}{2} & x>1 \end{cases}$$

comparison at $u \ll v$ gives[1]

$$A = \frac{1}{3}\int_0^\infty dx\, x^2 j_2(x) = \frac{\pi}{2}$$

Finally, we find

$$\int_0^\infty dx\, x j_1(ux) j_2(vx) = \frac{\pi u}{2v^3}\theta(v-u)$$

These all give

$$W_m = \frac{9}{\pi}\int_0^\infty ds\, e^{-st} j_1(\sqrt{s}) \int_0^\infty ds'\, e^{-s't} j_1(\sqrt{s'}) \Big[\frac{\pi}{3 s_>^{3/2}} + \frac{\pi}{3\sqrt{s}}\delta(s-s') - \frac{2\pi}{3s'\sqrt{s'}}\theta(s'-s)\Big]$$

$$= 3\int_0^\infty ds\, e^{-st} j_1(\sqrt{s}) \int_0^\infty ds'\, e^{-s't} j_1(\sqrt{s'}) \Big[\frac{1}{s_>^{3/2}} + \frac{1}{\sqrt{s}}\delta(s-s') - \frac{1}{s\sqrt{s}}\theta(s-s') - \frac{1}{s'\sqrt{s'}}\theta(s'-s)\Big]$$

where in the last line, we have used the symmetry of the first two integrals in s, s' to write the last term in a symmetric fashion. Now the last two terms cancel the first term and then give

$$W_m = 3\int_0^\infty ds\, e^{-st} j_1(\sqrt{s}) \int_0^\infty ds'\, e^{-s't} j_1(\sqrt{s'}) \frac{1}{\sqrt{s}}\delta(s-s')$$

$$= \boxed{3\int_0^\infty \frac{ds}{\sqrt{s}} e^{-2st} j_1^2(\sqrt{s})}$$

In $t \gg 1$, only the small s contribute to the integral and (using the Taylor series for j_1) we get

$$W_m \approx 3\int_0^\infty \frac{ds}{\sqrt{s}} e^{-2st}\frac{s}{9} = \frac{1}{3}\times(2t)^{-3/2}\times\Gamma(3/2) = \boxed{\frac{1}{12}\sqrt{\frac{\pi}{2}}t^{-3/2}}$$

d) Once again, in the integral for $\mathbf{A}$, only the small s matter. Keeping only the linear terms, we face the integral

$$\mathbf{A} \propto \int_0^\infty ds \sqrt{s} e^{-st} \boxed{\propto t^{-3/2}}$$

[1] I have used only the renormalized value of the definite integral setting $\sin\infty = \cos\infty = 0$.

Apparently, the vector potential is proportional to $t^{-3/2}$ *everywhere* at large t and the energy has to behave as t^{-3}. However, this apparent paradox is alleviated if we note that $s \ll \frac{1}{t}$ does not necessarily imply $\sqrt{s}r \ll 1$ and therefore we may need to keep the second Bessel function as it is. The $t^{-3/2}$ behavior is therefore valid only for

$$\boxed{r \lesssim \frac{1}{\sqrt{t}}}$$

While in the $r \gg 1/\sqrt{t}$ limit, the second Bessel function, ends the integral before the exponential deviates significantly from unity. Therefore, t is irrelevant and the vector potential remain constant. The *news* about the current being removed have not yet diffused to those distant points.

In summary, we find that some of the energy is decaying as t^{-3} while some of it remains constant. The whole budget is decaying as $t^{-3/2}$.

References

[1] J. K. Bloomfield, S. H. P. Face, Z. Moss, *Indefinite Integrals of Spherical Bessel Functions*, arXiv: 1703.06428

[2] R. Mehrem, "The Plane Wave Expansion, Infinite Integrals and Identities involving Spherical Bessel Functions," arXiv: 0909.0494