Electrodynamics III - HW#7

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1 Jackson; 6.4

a) In the steady state, there should be no force acting on a charged particle rotating around the z axis. This gives

$$\mathbf{E} = -\omega s\hat{\boldsymbol{\varphi}} \times \mathbf{B} = \omega s(B_s\hat{\mathbf{z}} - B_z\hat{\mathbf{s}})$$

And therefore the charge density is

$$\rho = \varepsilon\boldsymbol{\nabla}.\mathbf{E} = \varepsilon_0\omega s(\mu_0 J_\varphi - \frac{2}{s}B_z)$$
$$= \frac{\omega^2 s^2}{c^2}\rho - 2\varepsilon_0\omega B_z$$

which solves as

$$\rho = \frac{-2\varepsilon_0\omega B_z}{1-\frac{\omega^2 s^2}{c^2}}$$

From now on, we shall drop all terms of order ω^2 in comparison with first order corrections since they are all inevitably suppressed by $1/c^2$ factors (dimensional analysis). Therefore, ρ is given by the unperturbed (non-rotating) mgnetic field.

$$\rho \approx -2\varepsilon_0\omega B_z \approx -\frac{4\omega M}{3c^2} = \boxed{\frac{-m\omega}{\pi c^2 R^3}}$$

b, c) Integrating the internal electric field, we find that the electric potential on the surface of the sphere is given by

$$\Phi(R,\theta) = A + \frac{1}{2}\omega B_z R^2\sin^2\theta = A + \frac{\mu_0\omega m}{4\pi R}\sin^2\theta = -\frac{\mu_0\omega m}{6\pi R}P_2(\cos\theta)$$

Where in the last line, we have used the fact that the sphere has zero net charge to speculate that the potential will have no monopole terms. This immediately yields the potential outside the sphere to be

$$\Phi(r\geq R,\theta) = -\frac{\mu_0\omega m R^2}{6\pi r^3}P_2(\cos\theta)$$

This is clearly a pure quadrupole potential. With only a single non-vanishing component

$$q_{20} = -\sqrt{\frac{5}{\pi}}\frac{\omega m R^2}{3c^2}$$

Therefore

$$\boxed{Q_{33} = -\frac{4\omega m R^2}{3c^2}}$$

$$\boxed{Q_{12} = Q_{21} = Q_{13} = Q_{31} = Q_{23} = Q_{32} = 0\,,\quad Q_{11} = Q_{22}}$$

Finally, $\mathrm{Tr}\,Q = 0$ gives

$$\boxed{Q_{11} = Q_{22} = -Q_{33}/2}$$

The surface charge density is given by

$$\sigma(\theta) = \varepsilon_0(E_r^{out} - E_r^{in}) = \frac{m\omega}{2\pi R^2 c^2}\Big(- P_2(\cos\theta) + \sin^2\theta\Big)$$

$$= \boxed{\frac{m\omega}{3\pi R^2 c^2}\Big[1 - \frac{5}{2}P_2(\cos\theta)\Big]}$$

d) This is simply given by the potential difference between the north pole and the equator

$$\Phi(R,\pi/2) - \Phi(R,0) = -\frac{\mu_0\omega m}{6\pi R}\big[P_2(0) - P_2(1)\big] = \boxed{\frac{\mu_0 m\omega}{4\pi R}}$$

2 Jackson; 6.12

The admittance is given by

$$Y = \frac{1}{Z} = \frac{1}{R - iX} = \frac{R + iX}{R^2 + X^2} = G - iB$$

This gives

$$G = \frac{R}{R^2 + X^2}\,; \;\; B = \frac{-X}{R^2 + X^2}$$

a) Using $V = ZI$, we may write

$$\frac{1}{|V_i|^2}\int_V \sigma|\mathbf{E}|^2 dV = \frac{1}{R^2 + X^2}\frac{1}{|I_i|^2}\int_V \sigma|\mathbf{E}|^2 dV = \frac{R}{R^2 + X^2} = G \;\blacksquare$$

b) Using $V = ZI$, we may write

$$-\frac{4\omega}{|V_i|^2}\int_V (w_m - w_e) dV = \frac{-1}{R^2 + X^2}\frac{4\omega}{|I_i|^2}\int_V (w_m - w_e) dV = \frac{-X}{R^2 + X^2} = B \;\blacksquare$$

3 Jackson; 6.13

a) The relevant, phasor (harmonic time dependence in the form of $e^{-i\omega t}$) version of Maxwell's equations are

$$\boldsymbol{\nabla}.\mathbf{B} = \boldsymbol{\nabla}.\mathbf{E} = 0; \;\; \boldsymbol{\nabla}\times\mathbf{B} = \frac{-i\omega}{c^2}\mathbf{E}; \;\; \boldsymbol{\nabla}\times\mathbf{E} = i\omega\mathbf{B}$$

Considering a power expansion in ω as

$$\mathbf{E} = \sum_{n=0}^{\infty}\mathbf{E}_n\omega^n; \;\; \mathbf{B} = \sum_{n=0}^{\infty}\mathbf{B}_n\omega^n$$

we find the recursive equations

$$\boldsymbol{\nabla}\cdot\mathbf{B}_n = \boldsymbol{\nabla}\cdot\mathbf{E}_n = 0; \;\; \boldsymbol{\nabla}\times\mathbf{E}_n = i\mathbf{B}_{n-1}; \;\; \boldsymbol{\nabla}\times\mathbf{B}_n = \frac{-i}{c^2}\mathbf{E}_{n-1}$$

with the initial conditions

$$\boxed{\mathbf{E}_0 = \frac{Q}{\varepsilon_0 ab}(-\hat{\mathbf{z}})}; \quad \mathbf{B}_0 = 0$$

Where Q denotes the charge on each plate in the DC scenario. Here, the z axis is perpendicular to the capacitor plates, the x axis is aligned with the edges of length a and finally, the y axis is aligned with the edges of length b. The symmetry of the problem (ignoring fringing effects) asserts $\partial_y = 0$. The first recursive equation has the solution (considering symmetries and neglecting fringing effects)

$$\mathbf{B}_1 = (A + \frac{i\mu_0 Qx}{ab})\hat{\mathbf{y}}$$

At the free edge of the plane, the current is zero and therefore, we expect the magnetic field to vanish as well. This sets the constant A and gives

$$\boxed{\mathbf{B}_1 = \frac{i\mu_0 Q}{ab}(x-a)\hat{\mathbf{y}}}$$

Then, we can plug this in to the second equation and get

$$\mathbf{E}_2 = \Big[\frac{\mu_0 Q}{2ab}(x-a)^2 + B\Big]\hat{\mathbf{z}}$$

Once again, at the free edge, where the B field vanishes, we expect no corrections and therefore

$$\boxed{\mathbf{E}_2 = \frac{\mu_0 Q}{2ab}(x-a)^2\hat{\mathbf{z}}}$$

b) The current is given by $I = -i\omega Q(\omega)$ and therefore

$$|I|^2 = \omega^2 Q^2(\omega)$$

This time $Q(\omega)$ is the true charge on plates given by the Gauss law as

$$\begin{aligned} Q(\omega) &= -b\varepsilon_0 \sum_{n=0}^{\infty} \omega^n \int_0^a \mathbf{E}_n(x).\hat{\mathbf{z}}\, dx \\ &= Q\Big[1 - \frac{1}{6}(\frac{a\omega}{c})^2 + \cdots\Big] \end{aligned}$$

This gives the reactance as

$$\begin{aligned} X &= \frac{1}{\omega Q^2(\omega)} \int_V \Big(\frac{B^2}{\mu_0} - \varepsilon_0 E^2\Big) dV \\ &\approx \frac{bd}{\omega Q^2}\Big[\frac{\mu_0 Q^2 \omega^2 a}{3b^2} - \frac{Q^2}{\varepsilon_0 ab^2}\Big(1 - \frac{\omega^2 a^2}{3c^2}\Big)\Big]\Big[1 + \frac{1}{3}(\frac{a\omega}{c})^2\Big] \\ &\approx \omega\frac{\mu_0 ad}{3b} - \frac{1}{\omega}\frac{d}{\varepsilon_0 ab} = \omega L - \frac{1}{\omega C} \end{aligned}$$

where

$$\boxed{L = \frac{\mu_0 ad}{3b}; \quad C = \frac{\varepsilon_0 ab}{d}}$$

4 Jackson; 6.14

a) The recursive equations and the initial conditions are similar to the previous problem

$$\nabla \cdot \mathbf{E}_n = \nabla \cdot \mathbf{B}_n = 0\ ;\quad \nabla \times \mathbf{E}_n = i\mathbf{B}_{n-1}\ ;\quad \nabla \times \mathbf{B}_n = \frac{-i}{c^2}\mathbf{E}_{n-1}$$

$$\boxed{\mathbf{E}_0 = -\hat{\mathbf{z}}\frac{Q}{\pi\varepsilon_0 a^2}}\ ;\ \mathbf{B}_0 = 0$$

Then we get

$$\mathbf{B}_1 = \hat{\boldsymbol{\varphi}}\Big(\frac{A}{s} + \frac{i\mu_0 Q s}{2\pi a^2}\Big)$$

which is only bounded if

$$\boxed{\mathbf{B}_1 = \hat{\boldsymbol{\varphi}}\frac{i\mu_0 Q s}{2\pi a^2}}$$

Next, the electric field is corrected as

$$\mathbf{E}_2 = \Big(\frac{\mu_0 Q s^2}{4\pi a^2} + B\Big)\hat{\mathbf{z}}$$

Once again, in the absence of magnetic field corrections, we expect no electric field corrections and therefore

$$\boxed{\mathbf{E}_2 = \frac{\mu_0 Q s^2}{4\pi a^2}\hat{\mathbf{z}}}$$

To find the results Jackson demands in the next part, we also need to compute $\mathbf{B}_3$, this will be

$$\mathbf{B}_3 = \hat{\boldsymbol{\varphi}}\Big(\frac{C}{s} - \frac{i\mu_0 Q s^3}{16\pi a^2 c^2}\Big)$$

and after applying the boundary condition

$$\boxed{\mathbf{B}_3 = -\frac{i\mu_0 Q s^3}{16\pi a^2 c^2}\hat{\boldsymbol{\varphi}}}$$

b) First, we start by finding an expression for the current amplitude $|I| = \omega Q(\omega)$. To do this, we write

$$\begin{aligned} Q(\omega) &= -2\pi\varepsilon_0 \sum_{n=0}^{\infty} \omega^n \int_0^a s ds\, \mathbf{E}_n(s).\hat{\mathbf{z}} \\ &= Q\Big[1 - \frac{1}{8}\big(\frac{\omega a}{c}\big)^2 + \cdots\Big] \end{aligned}$$

Now we may write

$$\begin{aligned} \int_V w_e dV &\approx \frac{\varepsilon_0}{4} d \int_0^a 2\pi s ds\, \frac{Q^2}{\pi^2\varepsilon_0^2 a^4}\Big(1 - \frac{s^2\omega^2}{2c^2}\Big) \\ &= \frac{Q^2 d}{2\pi\varepsilon_0 a^4}\Big(\frac{a^2}{2} - \frac{\omega^2 a^4}{8c^2}\Big) \\ &= \frac{Q^2 d}{4\pi\varepsilon_0 a^2}\Big(1 - \frac{\omega^2 a^2}{4c^2}\Big) \\ &\approx \boxed{\frac{|I|^2 d}{4\pi\varepsilon_0 a^2 \omega^2}} \end{aligned}$$

and

$$\int_V w_m dV \approx \frac{d}{4\mu_0}\int_0^a 2\pi s ds\, \frac{\mu_0^2 Q^2 s^2}{4\pi^2 a^4}\omega^2\Big(1-\frac{s^2\omega^2}{4c^2}\Big)$$
$$= \frac{\mu_0 d Q^2\omega^2}{32\pi}\Big(1-\frac{\omega^2 a^2}{6c^2}\Big)$$
$$\approx \frac{\mu_0 d|I|^2}{32\pi}\Big(1+\frac{\omega^2 a^2}{4c^2}\Big)\Big(1-\frac{\omega^2 a^2}{6c^2}\Big)$$
$$\approx \boxed{\frac{\mu_0 d|I|^2}{32\pi}\Big(1+\frac{\omega^2 a^2}{12c^2}\Big)}$$

c) The reactance is

$$X = \frac{4\omega}{|I|^2}(W_m - W_e)$$
$$\approx \frac{\mu_0\omega d}{8\pi}\Big(1+\frac{\omega^2 a^2}{12c^2}\Big) - \frac{d}{\pi\varepsilon_0 a^2\omega} \approx \omega L - \frac{1}{\omega C}$$

where

$$\boxed{L \equiv \frac{\mu_0 d}{8\pi}\ ;\quad C \equiv \frac{\varepsilon_0\pi a^2}{d}}$$

The resonance frequency is

$$\omega_{\text{res.}} \equiv \frac{1}{\sqrt{LC}} \approx \boxed{\sqrt{8}\frac{c}{a}} \approx 2.83\frac{c}{a} > \beta_{01}\frac{c}{a} \approx 2.40\frac{c}{a}$$

5 Jackson; 7.7

a) The beam lasts for many periods and therefore we may consider the time dependence to be monochromatic, that is $e^{-i\omega t}$. Also, the same, relatively flat dependence on y, allows us to neglect the y dependence altogether. (A monotone dependence with zero frequency assumed.) Therefore, the incident beam may be written as

$$\mathbf{E}_i = \boldsymbol{\epsilon} e^{-i\omega t}\int d\theta\, A(\theta)\exp\Big[ik\Big(z\cos\theta\ +\ x\sin\theta\Big)\Big]$$

b) The reflected wave, has a phase shift and a reversed z-component of the wave vector.

$$\mathbf{E}_r = \boldsymbol{\epsilon} e^{-i\omega t}\int d\theta\, A(\theta)e^{i\phi(\theta)}\exp\Big[ik\Big(x\sin\theta\ -\ z\cos\theta\Big)\Big]$$

Now let us assume that $A(\theta)$ is narrowly concentrated around a specific angle

$$\theta_0 > \arcsin\frac{1}{n_r}$$

then using the taylor expansion of the sin and cos functions we have

$$\mathbf{E}_i = \boldsymbol{\epsilon} e^{i(\mathbf{k}.\mathbf{x}-\omega t)}\int d\theta\, A(\theta)\Big\{1 + i(\theta-\theta_0)\mathbf{k}_\perp.\mathbf{x} - \frac{(\theta-\theta_0)^2}{2}\Big[i\mathbf{k}.\mathbf{x} + (\mathbf{k}_\perp.\mathbf{x})^2\Big] + \cdots\Big\}$$

where

$$\mathbf{k} \equiv k(\cos\theta\hat{\mathbf{z}} + \sin\theta\hat{\mathbf{x}})\,;\ \ \mathbf{k}_\perp \equiv k(-\sin\theta\hat{\mathbf{z}} + \cos\theta\hat{\mathbf{x}})$$

Similarly, the reflected wave is

$$\mathbf{E}_r = \boldsymbol{\epsilon} e^{i(\mathbf{k}^r.\mathbf{x}-\omega t+\phi(\theta_0))}\int d\theta A(\theta)\Big\{1+i(\theta-\theta_0)\Big[\mathbf{k}^r_\perp.\mathbf{x}+\phi'(\theta_0)\Big]$$
$$-\frac{(\theta-\theta_0)^2}{2}\Big[i\mathbf{k}^r.\mathbf{x}+(\mathbf{k}^r_\perp.\mathbf{x})^2-2\phi'(\theta_0)\mathbf{k}^r_\perp.\mathbf{x}-i\phi''(\theta_0)+\phi'^2(\theta_0)\Big]+\cdots\Big\}$$

with

$$\mathbf{k}^r \equiv k(-\cos\theta\hat{\mathbf{z}}+\sin\theta\hat{\mathbf{x}})\,;\;\; \mathbf{k}^r_\perp \equiv k(\sin\theta\hat{\mathbf{z}}+\cos\theta\hat{\mathbf{x}})$$

Up to the linear term, this may be written as

$$\mathbf{E}_r(\mathbf{x},t) = \mathbf{E}_r^{(\text{geo})}(\mathbf{x}+\mathbf{D},t)$$

where $\mathbf{E}_r^{(\text{geo})}$ is the geometrically reflected wave in the absence of the extra phase $\phi(\theta)$; that is

$$\mathbf{E}_r^{(\text{geo})} \equiv \boldsymbol{\epsilon} e^{-i\omega t}\int d\theta\, A(\theta)\exp\Big[ik\Big(x\sin\theta\,-\,z\cos\theta\Big)\Big]$$

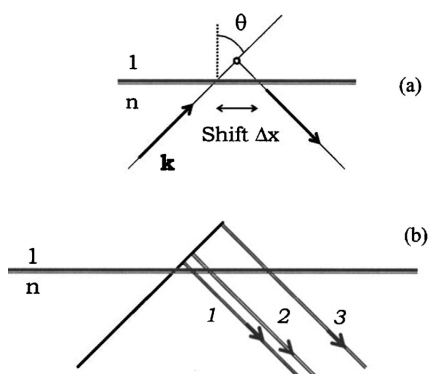
The spatial displacement satisfies

$$\begin{cases}\mathbf{k}^r.\mathbf{D} = \phi(\theta_0)\\ \mathbf{k}^r_\perp.\mathbf{D} = \phi'(\theta_0)\end{cases}$$

Therefore, the lateral displacement is given by

$$\boxed{D = \frac{\phi'(\theta_0)}{k}}$$

The direction of this lateral displacement is consistent with the following image if we find D to be negative. (Indeed we will!)



c) From Jackson's eq. 7.39, we have

$$e^{i\phi_\perp} = \frac{n\cos\theta - i\sqrt{n^2\sin^2\theta-1}}{n\cos\theta + i\sqrt{n^2\sin^2\theta-1}} \implies \phi_\perp = -2\arctan\frac{\sqrt{n^2\sin^2\theta-1}}{n\cos\theta}$$

This then gives

$$\boxed{D_\perp = -\frac{\lambda}{\pi}\frac{\sin\theta}{\sqrt{\sin^2\theta-1/n^2}}}$$

Similarly, using eq. 7.41, we get

$$e^{i\phi_{||}} = \frac{\cos\theta - in\sqrt{n^2\sin^2\theta-1}}{\cos\theta + in\sqrt{n^2\sin^2\theta-1}} \implies \phi_{||} = -2\arctan\frac{n\sqrt{n^2\sin^2\theta-1}}{\cos\theta}$$

which leads to

$$\boxed{D_{||} = \frac{D_\perp}{n^2\sin^2\theta-\cos^2\theta}}$$