

PSU Physics PhD Qualifying Exam Solutions
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Candidacy Exam
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Part I

1. A rigid pendulum consists of two perpendicular uniform co-planar rods of length $2a$ and $2b$, where $b > a$ (see figure below). The two rods have identical density. The pendulum is free to oscillate in the plane of the two rods about a pivot at point O .
- Find the equilibrium angle ϕ_0 (see Fig. 1(a)).
 - Suppose the pendulum is released from rest with the longer rod initially pointing vertically downward (see Fig. 1 (b)). During the subsequent oscillatory motion, find the maximum angle ϕ_m made by the longer rod with the vertical, expressing your answer in terms of the equilibrium angle ϕ_0 .

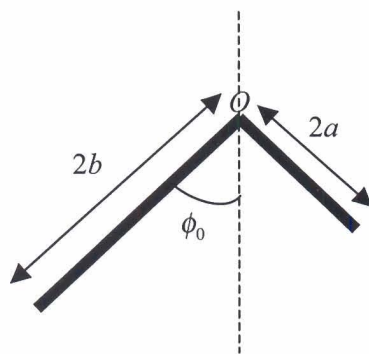


Fig. 1 (a)

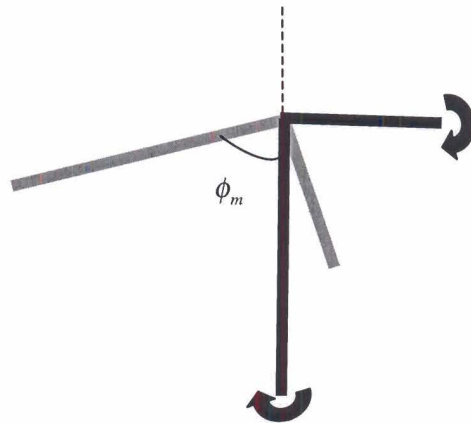
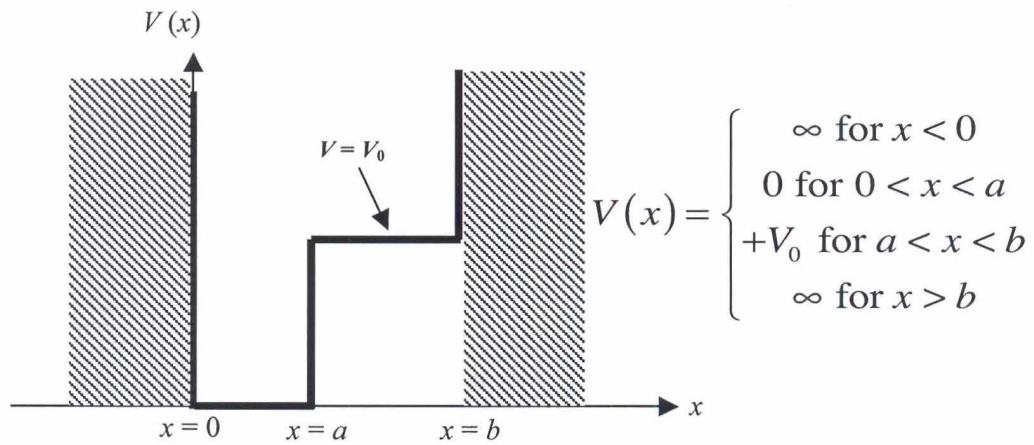


Fig. 1 (b)

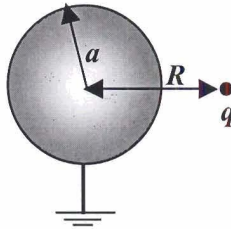
2. A particle of mass m is confined to the region $0 < x < b$ by the one dimensional potential defined below.



In addition, we are given the information that one of the eigenstates $\psi_n(x)$ has an eigenenergy $E_n = +V_0$. Your aim is to determine a relationship between the physical parameters a , b and V_0 that makes this possible.

- Solve the steady state Schrodinger equation for the eigenstate $\psi_n(x)$ in the region $0 < x < a$.
- Solve the steady state Schrodinger equation for the eigenstate $\psi_n(x)$ in the region $a < x < b$.
- Use continuity conditions to determine an equation that constrains the relationship between the physical parameters a , b and V_0 .

3. A small spherical plastic bead carries a uniformly distributed charge q . The bead is placed at a distance R from the center of a **grounded** spherical conductor of radius a ($a < R$) (see figure below). Your aim is to calculate the force on the bead due to the charge induced on the conductor.

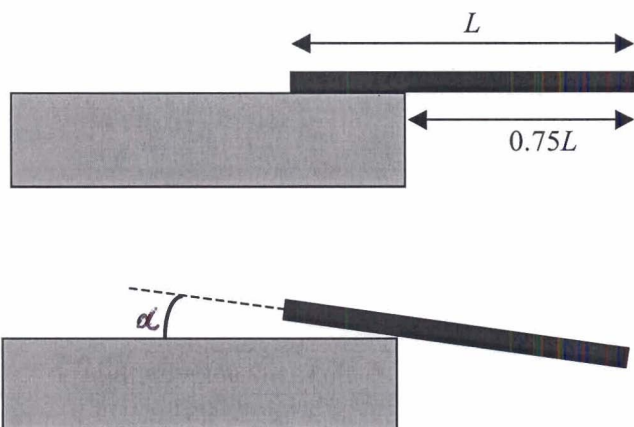


- a. It is possible to position a **single** image charge inside the spherical conductor such that it produces an equipotential surface with $V = 0$, identical to that of the grounded spherical conductor. Use strategic points on the equipotential surface to generate a pair of simultaneous equations involving the location and value of the image charge.
- b. Solve the equations in (a) for the location and value of the image charge, and hence determine the force on the bead.
4. N atoms of an ideal monatomic gas are enclosed in a bulb of volume V_0 that itself is placed inside an evacuated container of total volume V . This container has thermally insulating, rigid walls.
- a. The bulb suddenly breaks, allowing the gas to now occupy the entire container. Use a statistical physics argument to determine the change in entropy of the gas.
- b. Reconsider the above process from a thermodynamic viewpoint. Explain how to calculate the change in entropy even though the free expansion process is irreversible. Carry out this calculation and show that the answer agrees with the result you obtained in (a).

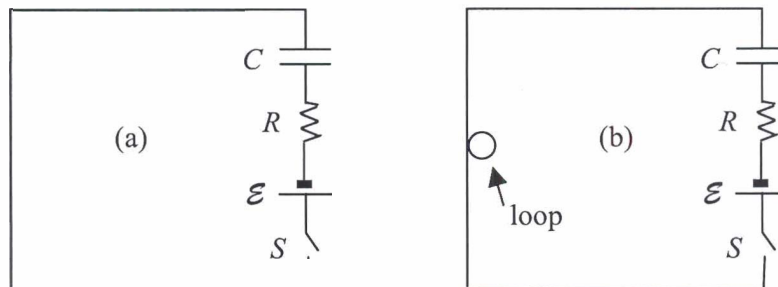
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Part II

1. A uniform stick is initially held horizontally with $\frac{3}{4}$ of its length sticking out over the edge of a table as shown in the upper figure below. It is then released and allowed to fall. The coefficient of static friction between the table and the stick is $\mu = 0.7$.
- a. Initially, the stick rotates about the point of support without slipping. Determine an expression for the force exerted by the table on the stick as a function of the angle α between the rod and the horizontal (see lower figure). You may assume that this force is always normal to the length of the rod. Your expression will include the (unknown) mass of the stick.
- b. Calculate the angle α_M at which the rod begins to slide.



2. Two point particles each of identical mass m but with opposite charges $+Q$ and $-Q$ are separated by a fixed distance d , forming a rigid electric dipole. The dipole is pivoted about the center of mass and constrained to rotate in a plane. The earth's gravitational field is not relevant in this problem.
- First, consider the system in the absence of any external electric field. Describe the system quantum mechanically, writing down the Hamiltonian, its eigenfunctions and eigenvalues. (Recall that the operator $L^2 = -\hbar^2 \frac{\partial^2}{\partial \phi^2}$, where L is the angular momentum.)
 - Suppose we now apply a uniform electric field \vec{E} in the plane of rotation. Write down the Hamiltonian for the system in terms of the angle of deviation ϕ between the dipole moment \vec{p} and the electric field \vec{E} .
 - Consider small oscillations of the electric dipole about the equilibrium position in the presence of the electric field. Rewrite the Hamiltonian in (b) for the case of small angles and show that it is essentially identical to the Hamiltonian of a quantum harmonic oscillator. Use this observation to determine an analytical expression for the ground state energy.
3. In both the RC circuits below, a capacitor $C = 1 \mu\text{F}$ and a resistor $R = 100 \text{ k}\Omega$ are connected in series to a battery of EMF $\mathcal{E} = 9 \text{ V}$. In both cases, switch S is initially open and is then closed at $t = 0$.



- Refer to figure (a) above: derive an expression for the time dependence of the current in the circuit once the switch is closed.
- Now, refer to figure (b) above: a circular loop of wire (radius $a = 5 \text{ mm}$, resistance $r = 5 \Omega$) is positioned close to -- but **electrically isolated** from -- a straight wire in the circuit. Determine an expression for the time-dependent current in the circular loop once the switch is closed. Assume that the loop is small enough to have negligible influence on the RC circuit and that the straight wire can be treated as being infinitely long.

In the questions that follow, **ignore Coulomb interactions between electrons.**

- a. Condensed matter physicists can now fabricate nanostructures with typical dimensions of ~ 10 nm and can study the behavior of small numbers of electrons confined within such nanostructures. Suppose you are given a semiconductor nanocube of side 10 nm, and you know that electrons confined to this nanocube behave as if they have an effective mass $m^* = 0.1 m_e$, where m_e is the free electron mass. Compute the total energy and the Fermi energy of **10 electrons** confined to the nanocube at $T = 0$ K. You may assume that inside the nanocube the potential $V = 0$ and outside the potential is infinite.

- b. Derive a general expression for the Fermi energy (at $T = 0$ K) of a three-dimensional free electron gas that consists of N electrons (where N is a very large number) confined to a macroscopic cube of side L .

I.1

a)

$$b^2 \sin \phi_0 = a^2 \cos \phi_0 \Rightarrow \boxed{\phi_0 = \arctan \frac{a^2}{b^2}}$$

b) The potential energies should match

$$-2\mu b^2 = -2\mu b^2 \cos \phi_m - 2\mu a^2 \sin \phi_m$$

therefore

$$\tan \phi_0 = \frac{a^2}{b^2} = \frac{1 - \cos \phi_m}{\sin \phi_m} = \frac{\sin \phi_m}{1 + \cos \phi_m} = \tan \frac{\phi_m}{2}$$

or

$$\boxed{\phi_m = 2\phi_0}$$

I.2

a) The Schrodinger equation $\psi_n'' + 2mV_0\psi_n = 0$ along with the boundary condition $\psi_n(0) = 0$ yields

$$\boxed{\psi_n(x) = A \sin(\sqrt{2mV_0}x)}$$

b) Here, we have $\psi_n'' = 0$ and therefore

$$\boxed{\psi_n = B(b - x)}$$

c) The continuity conditions are

$$\begin{cases} B(b - a) = A \sin(\sqrt{2mV_0}a) \\ -B = \sqrt{2mV_0}A \cos(\sqrt{2mV_0}a) \end{cases}$$

which can be met only if

$$\boxed{\sin(\sqrt{2mV_0}a) + \sqrt{2mV_0}(b - a) \cos(\sqrt{2mV_0}a) = 0}$$

I.3

a) Griffiths has proved that the image charge is a distance a^2/R from the center of the sphere and in the direction of the plastic bead. The value of the charge is $\boxed{q' = -aq/R}$.

b)

$$\boxed{F = \frac{aq^2}{4\pi\epsilon_0 R(R - a^2/R)^2} \text{ attractive}}$$

I.4

a) Each atom has now V/V_0 many times more options for its position. Therefore the system has $(V/V_0)^N$ times many more options for its spatial configuration. The change in entropy is then calculated as

$$\Delta S = \log \frac{\Omega}{\Omega_0} = \boxed{N \log(V/V_0)}$$

b) The energy, and therefore the temperature of the system does not change during the process. Therefore, we can calculate the entropy change as

$$\Delta S = \int \frac{dQ}{T} = N \int \frac{dU - dW}{pV} = N \int \frac{dV}{V} = \boxed{N \log(V/V_0)}$$

II.1

a) Conservation of energy implies

$$\frac{1}{2} I \dot{\theta}^2 = mg \frac{L}{4} \sin \theta$$

with

$$I = \int_{-L/4}^{+3L/4} x^2 \frac{m}{L} dx = \frac{7mL^2}{48}$$

This gives the angular velocity as

$$\dot{\theta}^2 = \frac{24g}{7L} \sin \theta$$

Using this and defining $\hat{\mathbf{m}}$ as the unit vector along the rod (right-wards) and $\hat{\mathbf{n}}$ as the unit vector normal to the rod (upwards), the hinge force is found as

$$\begin{aligned} \mathbf{F} &= m(\mathbf{a} - \mathbf{g}) = m \left(\frac{d^2}{dt^2} \frac{L}{4} \hat{\mathbf{m}} + g \cos \theta \hat{\mathbf{n}} - g \sin \theta \hat{\mathbf{m}} \right) \\ &= \boxed{mg \left(-\frac{13}{7} \sin \theta \hat{\mathbf{m}} + \frac{4}{7} \cos \theta \hat{\mathbf{n}} \right)} \end{aligned}$$

b) Just before slipping, the tangential to normal ratio of the hinge force equals the coefficient of friction.

$$0.7 = \frac{13 \sin \alpha_M}{4 \cos \alpha_M} \Rightarrow \boxed{\alpha_M = \arctan \left(\frac{2.8}{13} \right) \approx 12.2 \text{ deg}}$$

II.2

a) Starting from the Lagrangian

$$L = \frac{1}{4} m d^2 \dot{\varphi}^2$$

we find the Hamiltonian

$$\boxed{H = \frac{P^2}{md^2}}$$

Here $P = -i\partial_\varphi$ is conjugate to the angle coordinate φ . The eigenfunctions are

$$\psi_n = \frac{1}{\sqrt{2\pi}} e^{in\varphi}; \quad n = 0, \pm 1, \pm 2, \dots$$

with energies

$$E_n = \frac{n^2}{md^2}$$

b)

$$H = \frac{P^2}{md^2} - QEd \cos \varphi$$

c)

$$H \approx \frac{P^2}{2M} + \frac{1}{2}M\omega^2\varphi^2 - QEd$$

where

$$M \equiv \frac{1}{2}md^2 \quad \text{and} \quad \omega^2 \equiv \frac{2QE}{md}$$

The ground state energy is then found as

$$E_g \approx \frac{\omega}{2} - QEd = \sqrt{\frac{QE}{2md}} - QEd$$

II.3

a)

$$\text{KVL: } \mathcal{E} - \frac{q}{C} - R\dot{q} = 0$$

implies

$$q = C\mathcal{E}(1 - e^{-t/RC})$$

and in turn

$$I = \dot{q} = \frac{\mathcal{E}}{R} e^{-t/RC}$$

with

$$\frac{\mathcal{E}}{R} = 90 \mu A \quad \text{and} \quad RC = 100 ms$$

b) I assume the question is to find the loop current. I start with the flux

$$\Phi = \int_0^\pi \frac{\mu_0 I}{2\pi a(1 - \cos \theta)} \times 2a^2 \sin^2 \theta \, d\theta = \mu_0 I a \left(1 + \frac{2}{\pi}\right)$$

then

$$i = \frac{\dot{\Phi}}{r} = \frac{\mu_0 a \mathcal{E}}{R^2 C r} \left(1 + \frac{2}{\pi}\right) e^{-t/RC}$$

putting in the numbers one gets

$$i \approx 1.85 pA \times e^{-t/RC}$$

II.4

a) The 10 electrons fully fill the first 5 states. Two electrons end up in the $|111\rangle$ state; six end up in the $|211\rangle$, $|121\rangle$, and $|112\rangle$ states; finally, two end up somewhere in the subspace spanned by $|122\rangle$, $|212\rangle$, and $|221\rangle$. The total energy is therefore

$$U = \frac{\pi^2}{2m^*a^2}(2 \times 3 + 6 \times 6 + 2 \times 9) \approx \boxed{2.26 \text{ eV}}$$

The Fermi energy, is

$$E_F = \frac{9}{60}U \approx \boxed{338 \text{ meV}}$$

b)

$$N = \frac{L^3}{\pi^3} \times \frac{1}{8} \times \frac{4}{3}\pi K_F^3 \times 2$$

gives the largest occupied momentum as

$$K_F = \left(\frac{3\pi^2 N}{L^3}\right)^{1/3}$$

The Fermi energy is therefore

$$E_F = \frac{K_F^2}{2m^*} = \boxed{\left(\frac{3\pi^2 N}{L^3}\right)^{2/3} / 2m^*}$$