PSU Physics PhD Qualifying Exam Solutions
Fall 2006

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Candidacy Exam
Department of Physics
August 26, 2006

Part I

Instructions:

- The following problems are intended to probe your understanding of basic physical principles. When answering each question, indicate the principles being applied and any approximations required to arrive at your solution. *If information you need is not given, you may define a variable or make a reasonable physical estimate, as appropriate.* Your solutions will be evaluated based on clarity of physical reasoning, clarity of presentation, and accuracy.
- Please use a new blue book for each question. Remember to write your name and the problem number of the cover of each book.
- We suggest you read all *four* of the problems before beginning to work them. You should reserve time to attempt every problem.

Fundamental constants:

Avogadro's number	N_A	$6.022 \times 10^{23}\,\mathrm{mol}^{-1}$
Boltzmann's constant	k_B	$1.381 \times 10^{-23}\,\mathrm{J\,K}^{-1}$
Electron charge magnitude	e	$1.602 \times 10^{-19}\,\mathrm{C}$
Gas constant	R	$8.314\,\mathrm{J\,mol}^{-1}\,\mathrm{K}^{-1}$
Planck's constant	h	$6.626 \times 10^{-34}\,\mathrm{J\,s}$
	$\hbar = h/2\pi$	$1.055 \times 10^{-34}\,\mathrm{J\,s}$
Speed of light in vacuum	c	$2.998 \times 10^{8}\,\mathrm{m\,s}^{-1}$
Permittivity constant	ϵ_0	$8.854 \times 10^{-12}\,\mathrm{F\,m}^{-1}$
Permeability constant	μ_0	$1.257 \times 10^{-6}\,\mathrm{N\,A}^{-2}$
Gravitational constant	G	$6.674 \times 10^{-11}\,\mathrm{m}^3\,\mathrm{kg}^{-1}\,\mathrm{s}^{-2}$
Standard atmospheric pressure	1 atmosphere	$1.01 \times 10^{5}\,\mathrm{N\,m}^{-2}$
Stefan-Boltzmann constant	σ	$5.67 \times 10^{-8}\,\mathrm{W\,m}^{-2}\,\mathrm{K}^{-4}$
Electron rest mass	m_e	$9.109 \times 10^{-31}\,\mathrm{kg} = 0.5110\,\mathrm{MeV}\,c^{-2}$
Proton rest mass	m_p	$1.673 \times 10^{-27}\,\mathrm{kg} = 938.3\,\mathrm{MeV}\,c^{-2}$
Origin of temperature scales	$0\,^\circ\mathrm{C} = 273\,\mathrm{K}$	

I–1. A particle of mass m moves in a plane under the influence of a central attractive force

$$F = -\frac{K}{r^2} e^{-\mu r},$$

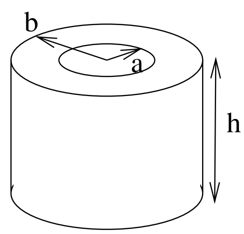
where K and μ are constants.

(a) Obtain an equation for the time dependence of r, the distance of the particle from the center of attraction.

(b) Determine the condition(s) on the constants such that circular motion of constant radius R is stable.

(c) Compute the frequency of small radial oscillations about this circular motion.

I–2. A hollow cylinder of inner radius a, outer radius b, and height h has a resistivity of ρ. What is the resistance $R = (V_b - V_a)/I$ for a current flowing from the outer to the inner radius.



I–3. The wave function of an electron in a one-dimensional infinite square well at time $t = 0$ is given by $\Psi(x, 0) = \sqrt{2/5}\psi_1(x) + \sqrt{3/5}\psi_2(x)$, where $\psi_1(x)$ and $\psi_2(x)$ are wave functions for the ground state and first excited stationary states of the system. [The well, of width a, extends from $x = 0$ to $x = a$, the energy eigenfunctions are $\psi_n(x) = \sqrt{2/a}\sin(n\pi x/a)$, and the eigenenergies are $E_n = n^2\pi^2\hbar^2/(2ma^2)$, where $n = 1, 2, 3, \ldots$.]

(a) Write down the wave function $\Psi(x, t)$ at time t in terms of $\psi_1(x)$ and $\psi_2(x)$.

(b) You measure the energy of an electron at time $t = 0$. Write down the possible values of the energy and the probability of measuring each.

(c) Calculate the expectation value of the energy in the state $\Psi(x, t)$ above.

I–4. Consider a collection of N independent particles, each bearing a magnetic moment μ which may be direction either up or down. Define n by saying that the number of spin-up moments is $(N + n)/2$, and the number of spin-down moments is $(N - n)/2$. The net magnetization is therefore $M = \mu n$. Assuming that there is no external magnetic field and that N is a very large number, derive the probability distribution $W(n)$ for having the magnetization $M = n\mu$. Derive the average value of the magentization, $\langle M \rangle$, and the average value of the squared magnetization, $\langle M^2 \rangle$.

Hint: Stirling's approximation may be useful: $\ln N! \simeq N \ln N - N$ for large N.

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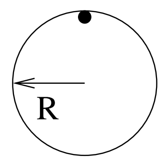
II–1. (a) State and prove the parallel axes theorem, which relates the moment of inertia of an object about an arbitrary axis to the moment of inertia about a parallel axis through the object's center of mass.

(b) A rigid loop of wire has radius R and is suspended from a frictionless pivot at its edge. Find the period of small oscillations under gravity. Neglect the radius of the wire and of the hook relative to R.



II–2. A magnetic field is generated by two coaxial plane coils, each of $n = 20$ turns of radius $a = 100\,\mathrm{mm}$, separated by distance a.

(a) Calculate the magnetic field at a point midway between the coils when a current $I = 2\,\mathrm{A}$ flows in each coil. The current in the two coils is parallel.

(b) Sketch the profile of the magnetic field strength summed along the axis of both coils.

II–3. (a) A spin-$\frac{1}{2}$ particle is in the presence of an external magnetic field applied in the z-direction, so that the relevant Hamiltonian is

$$H = -BS_z. \tag{II–1}$$

Which, if any, of the following are conserved quantities for this Hamiltonian: S^2, S_x, S_y, S_z?

(b) The normalized spin state of the particle at $t = 0$ is given by $|\Psi(t = 0)\rangle = a\,|\uparrow\rangle + b\,|\downarrow\rangle$, where $|\uparrow\rangle$ and $|\downarrow\rangle$ represent spin states with, respectively, spin parallel to and opposite to the z-axis, respectively. The coefficients a and b are real numbers that are arbitrary except for the normalization constraint. Calculate the expectation values at time t of (i) S_z and (ii) S_x.

II–4. The equilibrium separation between hydrogen atoms in molecular H_2 is $0.08\,\mathrm{nm}$, and the force constant of the bond is $580\,\mathrm{Nm}^{-1}$.

(a) Estimate the minimum energy (in Joules) to cause each molecule to (i) rotate, (ii) vibrate.

(b) Roughly sketch the dependence on temperature of the specific heat capacity of hydrogen gas between $30\,\mathrm{K}$ and $1000\,\mathrm{K}$.

I-1

a)

$$\boxed{\ddot{r} = \frac{l^2}{r^3} - \frac{K}{r}e^{-\mu r}}$$

where $l = r^2\dot{\theta}$ is the angular momentum per unit mass.

b) We need $\ddot{r} = 0$ which is equivalent to

$$\frac{\mu l^2}{K} = \mu R e^{-\mu R}$$

A small perturbation in the form of $r = R + x$ evolves according to

$$\ddot{x} = \Big(- \frac{3l^2}{R^4} + \frac{2K}{R^3}e^{-\mu R} + \frac{\mu K}{R^2}e^{-\mu R}\Big)x$$

which describes a stable oscillation if

$$\frac{3l^2}{R^4} > \frac{Ke^{-\mu R}}{R^2}\big(\frac{2}{R} + \mu\big)$$

$$\Leftrightarrow \frac{3l^2}{\mu K R^2} > e^{-\mu R}\big(1 + \frac{2}{\mu R}\big)$$

$$\Leftrightarrow \frac{3}{\mu R} > \frac{2}{\mu R} + 1$$

$$\Leftrightarrow \boxed{\mu R < 1}$$

c) This is already almost solved in the previous part

$$\omega^2 = \frac{3l^2}{R^4} - \frac{Ke^{-\mu R}}{R^2}\big(\frac{2}{R} + \mu\big) = \boxed{\frac{l^2}{R^4}(1 - \mu R)}$$

I-2

The potential is

$$\phi = V\frac{\log(s/a)}{\log(b/a)}$$

which gives

$$\mathbf{J} = \frac{-V\hat{\mathbf{s}}}{\log(b/a)s}$$

as for the current

$$I = \frac{2\pi h \sigma V}{\log(b/a)}$$

equivalent to a resistance

$$\boxed{R = \frac{\log(b/a)}{2\pi h \sigma}}$$

I-3

a)

$$\Psi(x,t) = \sqrt{\frac{2}{5}} \times \sqrt{\frac{2}{a}} \sin\Big(\frac{\pi x}{a}\Big) e^{-\frac{i\pi^2 t}{2ma^2}} + \sqrt{\frac{3}{5}} \times \sqrt{\frac{2}{a}} \sin\Big(\frac{2\pi x}{a}\Big) e^{-\frac{2i\pi^2 t}{ma^2}}$$

b) Using the Born rule and the given amplitudes

$$E = \begin{cases} \frac{\pi^2}{2ma^2} & \text{w.p. } \frac{2}{5} \\ \\ \frac{2\pi^2}{ma^2} & \text{w.p. } \frac{3}{5} \end{cases}$$

c)

$$\langle E \rangle = \frac{\pi^2}{2ma^2}(1 \times \frac{2}{5} + 4 \times \frac{3}{5}) = \boxed{\frac{7\pi^2}{5ma^2}}$$

I-4

In order to find the expectations, it is best to write the total magnetization as

$$M = \mu \sum_i X_i$$

where X_i are i.i.d. random variables distributed as

$$X_i = \begin{cases} +1 & \text{w.p. } \frac{1}{2} \\ \\ -1 & \text{w.p. } \frac{1}{2} \end{cases}$$

Then

$$\langle M \rangle = \mu \sum_i \langle X_i \rangle = \boxed{0}$$

and

$$\langle M^2 \rangle = \mu^2 \sum_{ij} \langle X_i X_j \rangle = \mu^2 N$$

For the distribution, we have

$$W(n) \equiv \mathbb{P}[M = n\mu] = 2^{-N} \binom{N}{\frac{N+n}{2}}$$

for large numbers, this is approximated as

$$\log W(n) = -N \log 2 + \log \Big(\frac{N!}{\frac{N+n}{2}! \, \frac{N-n}{2}!} \Big)$$

$$\approx -N \log 2 + N \log N - N - \frac{N+n}{2} \log \frac{N+n}{2} + \frac{N+n}{2} - \frac{N-n}{2} \log \frac{N-n}{2} + \frac{N-n}{2}$$

$$= N h\Big(\frac{N+n}{2N}\Big)$$

where

$$h(x) \equiv -x \log(x) - (1-x) \log(1-x)$$

II-1

a) Without loss of generality, assume that the center of mass is at the origin of coordinates. The moment of inertia around a point **a** is given by

$$I = \int |\boldsymbol{\mathscr{r}}_{\perp}|^2 dm = \int |\mathbf{r}_\perp - \mathbf{a}_\perp|^2 dm$$

$$= \int \left(r_\perp^2 - 2\mathbf{a}_\perp . \mathbf{r}_\perp + a_\perp^2\right) dm$$

$$= I_{CM} + Ma_\perp^2 - 2\mathbf{a}_\perp . \int \mathbf{r}_\perp dm = I_{CM} + Ma_\perp^2 \quad \blacksquare$$

b) In terms of the tilt angle θ, the kinetic and potential energies are as below

$$T = \frac{1}{2} I \dot{\theta}^2 = \frac{1}{2}(mR^2 + mR^2)\dot{\theta}^2 = mR^2\dot{\theta}^2$$

$$V \approx \frac{mgR}{2}\theta^2$$

comparison with a simple harmonic oscilator gives

$$\boxed{\omega^2 = \frac{g}{2R}}$$

II-2

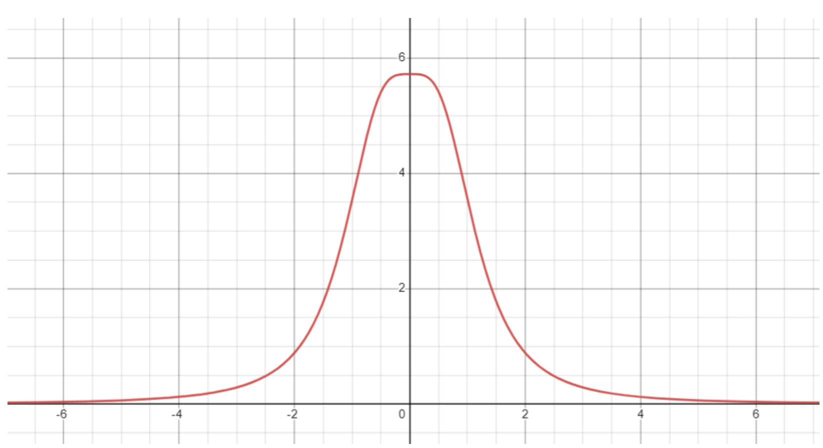
a) A simple application of the Bio-Savart formula yields the magnetic field as a function of z as

$$\mathbf{B} = \frac{n\mu_0 I}{4\pi a^2}\left\{\left[1 + (\frac{z}{a} + \frac{1}{2})^2\right]^{-3/2} + \left[1 + (\frac{z}{a} - \frac{1}{2})^2\right]^{-3/2}\right\}\hat{\mathbf{z}}$$

at $z = 0$:

$$\boxed{\mathbf{B} \approx 5.7 \times 10^{-4}\, T\, \hat{\mathbf{z}}}$$

b) In the following plot, the magnetic field is measured in Gauss and the length is measured in units of $a = 100mm$



II-3

a) A conserved operator must commute with H; therefore, $\boxed{\text{of the given list, only } S^2 \text{ and } S_z \text{ are conserved.}}$

b) S_z is conserved and equal to

$$\boxed{\langle S_z \rangle = \frac{1}{2}(a-b)}$$

For S_x

$$\langle S_x \rangle = \frac{1}{2}(2P_{+,x} - 1)$$

with

$$2P_{+,x} = \left| (1,1) \begin{pmatrix} ae^{iBt/2} \\ be^{-iBt/2} \end{pmatrix} \right|^2 = a^2 + b^2 - 2ab\cos(Bt)$$

Substituting this gives

$$\boxed{\langle S_x \rangle = \frac{1}{2}[2ab\cos(Bt) - 1]}$$

II-4

a) i) For rotation, the axis with the larger moment of inertia starts to rotate first. The first gap energy roughly determines the energy scale

$$\Delta E = \frac{3\hbar^2}{2I} \approx \frac{6\hbar^2}{m_H \ell^2} \approx \boxed{6.1 \times 10^{-21}\, J}$$

ii) This is simply given by the constant gap

$$\Delta E = \hbar\omega = \hbar\sqrt{\frac{K}{m_H}} \approx \boxed{6.1 \times 10^{-20}\, J}$$

b) Let us first convert the energy scales we found in the previous part to temperatures

$$T_{\text{rot.}} \approx 450\, K \quad ; \quad T_{\text{vib.}} \approx 4500\, K$$

The vibration is therefore not activated in this temperature range. The heat capacity starts from that of an ideal monatomic gas, i.e. $\frac{3}{2}Nk_B$ and slowly rises to $2Nk_B$ as the temperature goes from $T \ll T_{\text{rot.}}$ to $T \gg T_{\text{rot.}}$. A rough sketch is as below (ignore the $T < 30K$ and $T > 1000K$ part).