PSU Physics PhD Qualifying Exam Solutions
Spring 2006

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Candidacy Exam
Department of Physics
January 14, 2006

Part I

Instructions:

- The following problems are intended to probe your understanding of basic physical principles. When answering each question, indicate the principles being applied and any approximations required to arrive at your solution. *If information you need is not given, you may define a variable or make a reasonable physical estimate, as appropriate.* Your solutions will be evaluated based on clarity of physical reasoning, clarity of presentation, and accuracy.
- Please use a new blue book for each question. Remember to write your name and the problem number of the cover of each book.
- We suggest you read all *four* of the problems before beginning to work them. You should reserve time to attempt every problem.

Fundamental constants:

Avogadro's number	N_A	$6.022 \times 10^{23}\,\mathrm{mol}^{-1}$
Boltzmann's constant	k_B	$1.381 \times 10^{-23}\,\mathrm{J\,K}^{-1}$
Electron charge magnitude	e	$1.602 \times 10^{-19}\,\mathrm{C}$
Gas constant	R	$8.314\,\mathrm{J\,mol}^{-1}\,\mathrm{K}^{-1}$
Planck's constant	h	$6.626 \times 10^{-34}\,\mathrm{J\,s}$
	$\hbar = h/2\pi$	$1.055 \times 10^{-34}\,\mathrm{J\,s}$
Speed of light in vacuum	c	$2.998 \times 10^{8}\,\mathrm{m\,s}^{-1}$
Permittivity constant	ϵ_0	$8.854 \times 10^{-12}\,\mathrm{F\,m}^{-1}$
Permeability constant	μ_0	$1.257 \times 10^{-6}\,\mathrm{N\,A}^{-2}$
Gravitational constant	G	$6.674 \times 10^{-11}\,\mathrm{m}^3\,\mathrm{kg}^{-1}\,\mathrm{s}^{-2}$
Standard atmospheric pressure	1 atmosphere	$1.01 \times 10^{5}\,\mathrm{N\,m}^{-2}$
Stefan-Boltzmann constant	σ	$5.67 \times 10^{-8}\,\mathrm{W\,m}^{-2}\,\mathrm{K}^{-4}$
Electron rest mass	m_e	$9.109 \times 10^{-31}\,\mathrm{kg} = 0.5110\,\mathrm{MeV}\,c^{-2}$
Proton rest mass	m_p	$1.673 \times 10^{-27}\,\mathrm{kg} = 938.3\,\mathrm{MeV}\,c^{-2}$
Origin of temperature scales	$0\,^\circ\mathrm{C} = 273\,\mathrm{K}$	

I–1. A small mass m with initial velocity v goes by a star with a mass M at initial impact distance b. What is the distance of closest approach?

I–2. The space between the plates of thin parallel-plate capacitor is filled with a medium of conductivity σ and unit dielectric constant. The separation between the plates is d, and the plates are circular. A variable voltage $V = V_0 \sin \omega t$ is applied to the capacitor. Find the magnetic field inside the capacitor. Assume that the electric field between the plates is uniform, i.e., ignore edge effects. *In your calculation, assume that $\omega \ll c/L$ and $\omega \ll c^2\epsilon_0/(\sigma L^2)$, where L is maximum linear dimension of the capacitor.* What would affect your calculation if ω did not satisfy these conditions?

I–3. A paramagnetic solid is composed of spin-1/2 atoms (N per unit volume), each with a permanent magnetic dipole moment (μ per atom). In the presence of a magnetic field, of flux density B, the particles can occupy one of only two spin states, with the magnetic moments parallel or antiparallel to the B field, with energies $\pm\mu B$. The Boltzmann distribution tells us the number fN of dipoles oriented parallel and the number $(1 - f)N$ antiparallel. In the whole problem, ignore dipole-dipole interactions.

(a) Use Boltzmann statistics to determine the fraction of moments that point parallel to the field at temperature T. What is the net dipole moment per unit volume (or magnetization) $M(T)$.

(b) The solid is held at a temperature of 1 K in a magnetic field of 1 T and is thermally isolated and is in equilibrium. Next the magnetic field is reduced to 0.3 T. If no dipoles change their orientation (i.e., no further thermal fluctuations have had an effect), to what temperature does the final distribution of dipoles correspond?

I–4. An electron is constrained to move around a circular ring of radius R. Its quantum mechanical wave function is therefore a function of the polar angle around the ring: $\psi(\theta)$. There is a uniform external classical magnetic field B perpendicular to the ring.

(a) First at $B = 0$, find the energy eigenstates and eigenfunctions. What is the ground state energy?

(b) Repeat with $B \neq 0$. (You will probably find it useful to obtain a vector potential for the magnetic field.) Show that the ground state is doubly degenerate when the magnetic flux through the ring is a half-integer multiple of ϕ_0. Here ϕ_0 is the elementary unit of magnetic flux, which is hc/e or h/e depending on the system of units.

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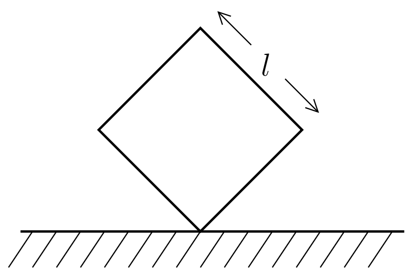
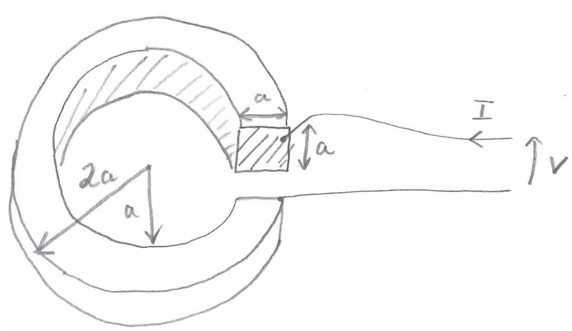
II–1. A homogeneous cube, each edge of which has a length l and whose mass is M, is initially in a position of unstable equilibrium with one edge in contact with a horizontal plane. The cube is then given a very small displacement and allowed to fall. Find the moment of inertia about a line that is through the center of mass and parallel to one of the edges of the cube. Then find the angular velocity of the cube when one face strikes the plane in the two cases: (i) the edge cannot slide on the plane, (ii) the edge can slide without friction.



II–2. Consider an object in the shape of a split ring. It is a circular ring with a square cross section, with a very small slit opened in it, and it is made of an electrically conducting material of resistivity ρ. The inner and outer radii of the ring are a and $2a$, and the thickness is a. Wires of negligible resistance are connected to the exposed faces.

Derive an expression, in terms of ρ and a, for the resistance R of the split ring as measured between the two wires.



II–3. (a) Define the term "chemical potential".

(b) What steps would you go through to calculate the chemical potential of a classical monatomic ideal gas, given its temperature, volume and the number of atoms? *There is no need to perform the calculation. Just explain clearly the steps needed to do the calculation.*

(c) In a container of such a gas is placed a solid on whose surface the atoms of the gas can be adsorbed. The adsorbed atoms form a two-dimensional ideal gas, with the energy of one atom being $\mathbf{p}^2/(2m) - \epsilon_0$, where $\mathbf{p}$ is its (two-component) momentum vector and ϵ_0 is the binding energy which holds the atom on the surface.

List the steps you would go through to obtain the number n' of atoms adsorbed per unit area of the solid's surface when the pressure of the surrounding gas is P and the temperature is T, and the system is in thermodynamic equilibrium.

II–4. The x-component of the spin of a spin 1 particle is measured, with the result $S_x = 0$. A second measurement is made, now of S_z. Find the possible values from the measurement and the corresponding probabilities.

I-1

Conservation of energy and angular momentum (per unit mass) read

$$\frac{v^2}{2} = -\frac{GM}{a} + \frac{u^2}{2}$$

$$bv = au$$

These combine to yield the minimum distance a as

$$\boxed{a = -\frac{GM}{v^2} + \sqrt{(\frac{GM}{v^2})^2 + b^2}}$$

I-2

From the symmetries and linearity of Maxwell's equations, we find the field forms as

$$\mathbf{E} = \mathrm{Re}\left[E(s)e^{-i\omega t}\right]\hat{\mathbf{z}}\ ; \quad \mathbf{B} = \mathrm{Re}\left[B(s)e^{-i\omega t}\right]\hat{\boldsymbol{\varphi}}$$

Maxwell's equations then read

$$\begin{cases} \frac{dE}{ds} = -i\omega B \\ \\ \frac{dB}{ds} = \mu_0(\sigma - i\omega\varepsilon_0)E \end{cases}$$

As we will see, the approximations allow us to use a uniform electric field

$$E^{(0)}(s) \approx \frac{-iV_0}{d}$$

when computing the magnetic field. This gives

$$B^{(1)}(s) \approx \frac{-iV_0}{d}\mu_0(\sigma - i\omega\varepsilon_0)s$$

or

$$\boxed{\mathbf{B} \approx \frac{\mu_0 V_0 s}{d}\hat{\boldsymbol{\varphi}}\Big[\sigma\sin(\omega t)\ -\ \omega\varepsilon_0\cos(\omega t)\Big]}$$

To see how good the uniform field approximation is, we can find the first correction to the electric field as

$$E^{(2)} = -\frac{\omega V_0 \mu_0}{2d}(\sigma - i\omega\varepsilon_0)s^2$$

For the approximation to hold, we need the following to be small

$$\frac{|E^{(2)}|}{|E^{(0)}|} \le \frac{\omega}{2}\mu_0 L^2|(\sigma - i\omega\varepsilon_0)| \le \frac{1}{2}\Big(\frac{\omega L}{c}\Big)^2 + \frac{1}{2}\omega\mu_0 L^2\sigma$$

which is guaranteed by the assumption.

I-3

a)

$$\frac{1-f}{f} = e^{-2\beta\mu B} \quad \Rightarrow \quad \boxed{f = \frac{1}{1+e^{-2\beta\mu B}}}$$

$$\mathbf{M} = n\mu(2f-1)\hat{\mathbf{B}} = n\mu \tanh(\beta\mu B)\hat{\mathbf{B}}$$

b)

$$n\mu \tanh(\beta\mu B) = n\mu \tanh(\beta'\mu B') \quad \Rightarrow \quad \boxed{T = 0.3\,K}$$

I-4

a)

$$L = \frac{1}{2}mR^2\dot{\theta}^2 \quad \Rightarrow \quad H = \frac{P^2}{2mR^2}$$

The energy eigen states and eigenvalues are given by

$$\boxed{\psi_n = \frac{1}{\sqrt{2\pi}}e^{in\theta}} \;\; ; \;\; \boxed{E_n = \frac{n^2}{2mR^2}}$$

for all $n \in \mathbb{Z}$. The ground state corresponds to $n = 0$:

$$\boxed{\psi_0 = \frac{1}{\sqrt{2\pi}} \; ; \; E_0 = 0}$$

b) This time we need to include the vector potential

$$\mathbf{A} = \frac{1}{2}Bs\hat{\boldsymbol{\varphi}}$$

in the Lagrangian

$$L = \frac{1}{2}mR^2\dot{\theta}^2 - \frac{1}{2}eBR^2\dot{\theta}$$

which leads to a Hamiltonian

$$H = \frac{\left(P + \frac{1}{2}eBR^2\right)}{2mR^2}$$

The eigen states are not changed

$$\boxed{\psi_n = \frac{1}{\sqrt{2\pi}}e^{in\theta}}$$

but the energies become

$$\boxed{E_n = \frac{\left(n + \frac{1}{2}eBR^2\right)}{2mR^2}}$$

This is degenerate if and only if

$$\frac{eBR^2}{2} = \ell \in \mathbb{Z}$$

which yields the magnetic flux as

$$\Phi = \pi R^2 B = \ell\Phi_0$$

II-1

$$I = M\langle x^2 + y^2\rangle = 2Ml^2\langle\big(\frac{x}{l}\big)^2\rangle = 2Ml^2\int_{-\frac{1}{2}}^{+\frac{1}{2}} x^2\,dx = \boxed{\frac{1}{6}Ml^2}$$

i)

$$I' = \frac{1}{6}Ml^2 + \frac{1}{4}Ml^2 = \frac{5}{12}Ml^2$$

is the moment of inertial around the constant pivot. Conservation of energy reads

$$Mgl(\frac{1}{\sqrt{2}} - \frac{1}{2}) = \frac{5}{24}Ml^2\omega_{\text{i}}^2$$

which leads to

$$\boxed{\omega_{\text{i}} = \sqrt{\frac{12(2-\sqrt{2})g}{5\sqrt{2}l}}}$$

ii) This time, the conservation of energy reads

$$Mgl(\frac{1}{\sqrt{2}} - \frac{1}{2}) = \frac{1}{12}Ml^2\omega_{\text{ii}}^2 + \frac{1}{2}MV^2$$

where

$$V = \frac{l\omega_{\text{ii}}}{\sqrt{2}}$$

is the velocity of the center of mass of the block. The angular velocity is finally found to be

$$\boxed{\omega_{\text{ii}} = \sqrt{\frac{3(2-\sqrt{2})g}{2\sqrt{2}l}}}$$

II-2

In the steady state, the potential ϕ satisfies the Laplace equation and the mixed Dirichlet and Neumann boundary conditions. (Fixed potential where the wires are touching and vanishing normal derivative elsewhere) Therefore, we can guess the solution by virtue of uniqueness:

$$\phi = \frac{V\varphi}{2\pi}$$

This leads to

$$\mathbf{J} = -\sigma\boldsymbol{\nabla}\phi = -\frac{\sigma V}{2\pi s}\hat{\boldsymbol{\varphi}}$$

Therefore the current from high potential to low potential is

$$I = \frac{V}{R} = \frac{\sigma V}{2\pi}\int\frac{dA}{s} = \frac{\sigma a V}{2\pi}\log(2)$$

which is equivalent to

$$\boxed{R = \frac{2\pi}{\sigma a\log(2)}}$$

II-3

a) One can use the infinitesimal form of the first law

$$dE = -pdV + TdS + \mu dN$$

to define the chemical potential as

$$\mu = \left(\frac{\partial E}{\partial N}\right)_{V,S}$$

This is also equivalent to

$$\mu = \left(\frac{\partial G}{\partial N}\right)_{p,T}$$

b) First, I should compute the partition function

$$Z(T,V,N) \equiv \int \frac{e^{-\beta H}}{(2\pi)^{3N}}\, d^{3N}\mathbf{p}\, d^{3N}\mathbf{x}$$

then, use

$$Z = e^{-\beta F}$$

to find the Free energy $F = E - TS$ in terms of (T,V,N). Finally, I should use

$$\mu = \left(\frac{\partial F}{\partial N}\right)_{V,T}$$

to find $\mu(T,V,N)$.

c) For such an open system, a condition for the equilibrium is the equality of chemical potentials. So we need to repeat the procedure for the 2D gas to find $\mu(T,A,N)$. Use the equation of state

$$p = nkT$$

where n is the number density in any dimension to express μ in terms of p,T for the 3D gas and then finally write

$$\mu_{3D}(P,T) = \mu_{2D}(T,A,n'A)$$

to find n'.

II-4

Using

$$J_{\pm}\left|j,m\right\rangle = \sqrt{j(j+1) - m(m\pm 1)}\left|j,m\pm 1\right\rangle$$

we find the J_+ operator as

$$J_+ = \begin{pmatrix} & \sqrt{2} & \\ & & \sqrt{2} \\ & & \end{pmatrix}$$

then

$$J_x = \frac{J_+ + J_-}{2} = \frac{J_+ + J_+^\dagger}{2} = \frac{1}{\sqrt{2}}\begin{pmatrix} & 1 & \\ 1 & & 1 \\ & 1 & \end{pmatrix}$$

Now after the measurement, we have $J_x\left|\psi\right\rangle = 0$. Apart from a total phase, this means

$$\left|\psi\right\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

Now it is fairly easy to see that the J_z measurement will result in

$$\boxed{J_z = \begin{cases} +1 & \text{w.p. } \frac{1}{2} \\ \\ -1 & \text{w.p. } \frac{1}{2} \end{cases}}$$