PSU Physics PhD Qualifying Exam Solutions
Spring 2007

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Candidacy Exam
Department of Physics
January 20, 2007

Part I

Instructions:

- The following problems are intended to probe your understanding of basic physical principles. When answering each question, indicate the principles being applied and any approximations required to arrive at your solution. *If information you need is not given, you may define a variable or make a reasonable physical estimate, as appropriate.* Your solutions will be evaluated based on clarity of physical reasoning, clarity of presentation, and accuracy.
- Please use a new blue book for each question. Remember to write your name and the problem number of the cover of each book.
- We suggest you read all *four* of the problems before beginning to work them. You should reserve time to attempt every problem.

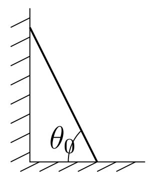
Fundamental constants:

Avogadro's number	N_A	$6.022 \times 10^{23}\,\mathrm{mol}^{-1}$
Boltzmann's constant	k_B	$1.381 \times 10^{-23}\,\mathrm{J\,K}^{-1}$
Electron charge magnitude	e	$1.602 \times 10^{-19}\,\mathrm{C}$
Gas constant	R	$8.314\,\mathrm{J\,mol}^{-1}\,\mathrm{K}^{-1}$
Planck's constant	h $\hbar = h/2\pi$	$6.626 \times 10^{-34}\,\mathrm{J\,s}$ $1.055 \times 10^{-34}\,\mathrm{J\,s}$
Speed of light in vacuum	c	$2.998 \times 10^{8}\,\mathrm{m\,s}^{-1}$
Permittivity constant	ϵ_0	$8.854 \times 10^{-12}\,\mathrm{F\,m}^{-1}$
Permeability constant	μ_0	$1.257 \times 10^{-6}\,\mathrm{N\,A}^{-2}$
Gravitational constant	G	$6.674 \times 10^{-11}\,\mathrm{m}^3\,\mathrm{kg}^{-1}\,\mathrm{s}^{-2}$
Standard atmospheric pressure	1 atmosphere	$1.01 \times 10^{5}\,\mathrm{N\,m}^{-2}$
Stefan-Boltzmann constant	σ	$5.67 \times 10^{-8}\,\mathrm{W\,m}^{-2}\,\mathrm{K}^{-4}$
Electron rest mass	m_e	$9.109 \times 10^{-31}\,\mathrm{kg} = 0.5110\,\mathrm{MeV}\,c^{-2}$
Proton rest mass	m_p	$1.673 \times 10^{-27}\,\mathrm{kg} = 938.3\,\mathrm{MeV}\,c^{-2}$
Origin of temperature scales	$0\,^\circ\mathrm{C} = 273\,\mathrm{K}$	

I–1. A uniform ladder of mass m and length l leans against a frictionless vertical wall and rests on a frictionless horizontal floor. It is released from rest, with the ladder and the floor initially making an angle θ_0.

(a) Derive the moment of inertia of the ladder about its center-of-mass in the plane in which the ladder is moving.

(b) At some point, the ladder will separate from the *wall*. Determine the angle the ladder makes with the floor when this happens.



I–2. A perfectly insulating hollow sphere of radius R has a charge Q uniformly spread on its surface. It spins with angular velocity ω. Find the magnetic dipole moment.

I–3. Derive the following equation[1] for the time-dependence of the expectation value of a (time-independent) quantum mechanical operator V:

$$\frac{d}{dt}\langle\psi|V|\psi\rangle = \frac{1}{i\hbar}\langle\psi|\,[V,H]\,|\psi\rangle\,, \tag{I–1}$$

where H is the Hamiltonian.

Consider now an electron immersed in a uniform homogeneous field along the z-axis. This gives the following term in the Hamiltonian:

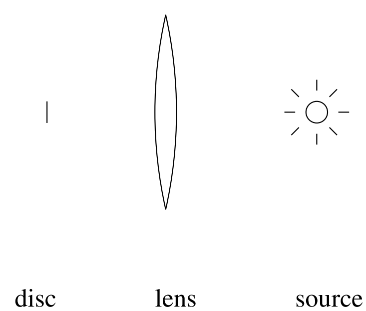
$$\frac{Be}{m}\,S_z, \tag{I–2}$$

where B is the magnetic field, S_z is the z-component of the electron's spin operator, m is its mass, and e is the absolute value of the electron's charge. Assume that the remainder of the Hamiltonian is spin-independent.

[1] Note that there was a sign error in this equation on the exam as given; it has been corrected here.

Suppose that, at time $t = 0$, the electron is in a state with its spin component along the x-axis being $\hbar/2$. Determine the expectation values of the x, y and z components of the electron's spin operators, $\langle\psi|S_\alpha|\psi\rangle$, at any later time. Qualitatively describe the time-dependence of the vector $\langle\psi|\vec{S}|\psi\rangle$.

I–4. The diagram shows a spherical source of light illuminating a lens which focuses radiation onto a thin black heat-conducting disk. The light source is 1 mm in diameter and emits 100 W of radiation isotropically. The lens is 2 cm in diameter, has a focal length of 10 cm, and transmits all the radiation that impinges on it. The diameter of the disk is 0.5 mm and the image of the light source is at the disk and exactly covers the disk. Determine the equilibrium temperature of the disk. (See the table of constants for the value of the Stefan-Boltzmann constant.)



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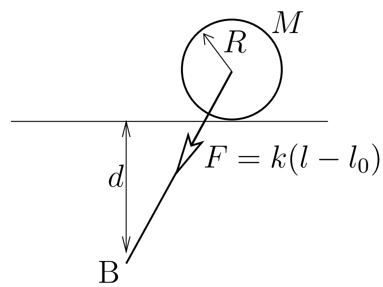
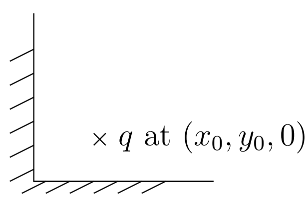
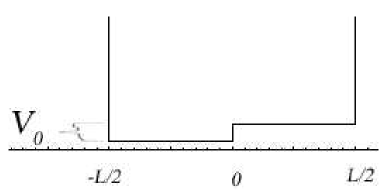
II–1. A azimuthally symmetric disk of radius R and mass M rolls without slipping on a horizontal surface. The momenta of inertia of the disk about its center is I, and a spring/springs are attached between the axis and a point B a distance d below the plane. The spring(s) have a length l_0 when relaxed and the force exerted by the springs towards B is $k(l - l_0)$, where l is the distance between the center of the disk and the point of attraction. Find the frequency of small oscillations about the position of equilibrium. All motion is in a plane perpendicular to the horizontal surface.



II–2. The region of space $(x > 0, y > 0)$ is bound on two sides by grounded conducting planes. A charge q is placed in this region at the position $(x_0, y_0, 0)$. Calculate the work required to move the charge from $(x_0, y_0, 0)$ to $(\infty, y_0, 0)$.



II–3. A particle of mass M is bound in a 1D infinite potential well of width L that is divided into two equal parts:



On the left side, the potential is zero, on the right side the potential is V_0. Everywhere else the potential is infinite. The initial wave function for the particle is given by the form

$$\psi(x) = \begin{cases} A_1 \sin(8\pi x/L) & \text{for } -\dfrac{L}{2} < x < 0, \\ A_2 \sin(4\pi x/L) & \text{for } 0 < x < \dfrac{L}{2}. \end{cases} \tag{II–1}$$

(a) Under what condition(s) on V_0, M, and L is this an eigenfunction of energy for the 1D Schrödinger equation?

(b) Determine values for A_1 and A_2 that give a normalized energy eigenfunction of the given form.

(c) What is its energy?

(d) Now assume that the two halves are suddenly separated to form two separate infinite wells, each of width $L/2$. (Imagine suddenly inserting an infinite delta-function potential barrier at the origin to separate the halves, $V(x) \mapsto V(x) + \infty\delta(x)$.) Determine the probabilities that the particle ends up in either separated well.

II–4. An isolated impurity site in a semiconductor is in thermal equilibrium with a reservoir at a chemical potential μ and temperature T. It is modeled as follows: Up to two extra electrons can bind to the impurity. Each electron is the same spatial state (an s-wave orbital). The energy of a one electron state is E and the energy of a two-electron state is $2E - V$.

(a) List the four possible electronic states in a standard basis of spin eigenstates.

(b) Calculate the grand partition function for electrons at the impurity site and calculate the mean number $\langle n \rangle$ of electrons.

(c) For what value of μ is $\langle n \rangle = 1$?

(d) An external magnetic field B is now applied, and interacts with electron spin in the usual way ("Zeeman term"). Calculate the spin susceptibility when $\langle n \rangle = 1$. Interpret the results when $V \ll k_{\mathrm{B}}T$ and when $V \gg k_{\mathrm{B}}T$.

I-1

a)

$$I = \int_{-\frac{l}{2}}^{+\frac{l}{2}} x^2 \, \frac{m}{l} dx \, = \frac{1}{12} ml^2$$

b) The center of mass moves on a circle of radius $\frac{l}{2}$ and centered at the corner. Therefore, the kinetic energy is

$$T = \frac{1}{2} m (\frac{l}{2})^2 \dot{\theta}^2 + \frac{1}{2} I \dot{\theta}^2 = \frac{1}{6} ml^2 \dot{\theta}^2$$

therefore the conservation of energy gives

$$\frac{1}{6} ml^2 \dot{\theta}^2 = mg \frac{l}{2} (\sin\theta_0 - \sin\theta) \;\Rightarrow\; \dot{\theta} \propto \big(\sin\theta_0 - \sin\theta\big)^{\frac{1}{2}}$$

this means the horizontal component of the linear momentum satisfies

$$P_x \propto \sin\theta \big(\sin\theta_0 - \sin\theta\big)^{\frac{1}{2}}$$

At the point of separation, P_x attains its maximum; the first order condition gives

$$\sin\theta^* = \frac{2}{3} \sin\theta_0$$

I-2

We just need to sum the magnetic dipole for each current loop

$$\begin{aligned} \boldsymbol{\mu} = \int \mathbf{A} dI &= \int_0^{\pi} \big(\pi R^2 \sin^2\theta \, \hat{\mathbf{z}}\big) \frac{\omega}{2\pi} \sigma R^2 d\theta \, \sin\theta \, (2\pi) \\ &= \boxed{(\frac{4}{3}\pi R^3) \sigma R \omega \hat{\mathbf{z}}} \end{aligned}$$

I-3

The Schrödinger equation is

$$\left| \dot{\psi} \right\rangle = -iH \left| \psi \right\rangle$$

therefore

$$\begin{aligned} \frac{d}{dt} \left\langle \psi \right| V \left| \psi \right\rangle &= \left\langle \dot{\psi} \right| V \left| \psi \right\rangle + \left\langle \psi \right| V \left| \dot{\psi} \right\rangle \\ &= -i \left\langle \psi \right| [V, H] \left| \psi \right\rangle \quad \blacksquare \end{aligned}$$

For the spin components, we have

$$\frac{d}{dt} \langle \mathbf{S} \rangle = -\frac{iBe}{m} \langle [\mathbf{S}, S_z] \rangle = \frac{eB}{m} \hat{\mathbf{z}} \times \langle \mathbf{S} \rangle$$

Given the initial condition $\langle \mathbf{S} \rangle = \frac{1}{2} \hat{\mathbf{x}}$, we have

$$\boxed{\langle \mathbf{S} \rangle = \frac{1}{2} \big(\hat{\mathbf{x}} \cos \frac{eBt}{m} + \hat{\mathbf{y}} \sin \frac{eBt}{m}\big)}$$

This is a rotating vector in the $x - y$ plane.

I-4

To find the power that reaches the disk, we need to find the distances. If p and q are the distances from the lens to the source and the disk respectively and $f = 10cm$ is the focal distance of the lens, then

$$\frac{1}{f} = \frac{1}{p} + \frac{1}{q}$$

from the size ratio of the image and the object we also have $p = 2q$. This yields

$$p = 30cm \; ; \; q = 15cm$$

Therefore the fraction of the power that reaches the lens is given by $\sin^2 \frac{\theta}{2}$ where $\tan\theta = \frac{1}{30}$. This is equivalent to

$$P_{\text{Disk}} = 100W \times \frac{1}{2}(1 - \frac{1}{\sqrt{1 + \frac{1}{30^2}}}) \approx 28\,mW$$

In equilibrium, this is also the radiation power *from* the disk

$$28\,mW \approx 2 \times \sigma T^4 \times \pi \frac{(0.5\,mm)}{4} \quad \Rightarrow \quad \boxed{T \approx 1100\,K}$$

II-1

If x denotes the horizontal displacement of the disk, then

$$T = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}I\frac{\dot{x}^2}{R^2} = \frac{\dot{x}^2}{2}\Big(M + \frac{I}{R^2}\Big)$$

$$V = \frac{1}{2}k\Big(\sqrt{x^2 + (R+d)^2} - l_0\Big)^2 \approx \frac{x^2}{2} \times k\Big(1 - \frac{l_0}{R+d}\Big) + cte.$$

Therefore the small oscillation frequency is

$$\boxed{\omega^2 = \frac{k}{m}\frac{1 - l_0/(R+d)}{1 + I/MR^2}}$$

II-2

Using the 3 image charges, the $x-$component of the electrostatic force is found as

$$F_x = \frac{q^2}{16\pi\varepsilon_0}\Big(\frac{x}{r^3} - \frac{1}{x^2}\Big)$$

Integration gives

$$W = -\int_{(x_0,y_0)}^{(\infty.y_0)} F_x dx = \boxed{\frac{q^2}{16\pi\varepsilon_0}\Big(\frac{1}{x_0} - \frac{1}{\sqrt{x_0^2 + y_0^2}}\Big)}$$

II-3

a) The energies on both sides should match

$$\frac{1}{2m}(\frac{8\pi}{L})^2 = \frac{1}{2m}(\frac{4\pi}{L})^2 + V_0$$

or, equivalently

$$\boxed{V_0 = \frac{24\pi^2}{mL^2}}$$

b) Continuity of the slope implies $A_2 = 2A_1$, and normalization implies

$$1 = \frac{A_1^2}{2} \times \frac{L}{2} + \frac{A_2^2}{2} \times \frac{L}{2}$$

Combined, these yield

$$\boxed{A_1 = \frac{2}{\sqrt{5L}}} ; \boxed{A_2 = \frac{4}{\sqrt{5L}}}$$

c)

$$\boxed{E = \frac{32\pi^2}{mL^2}}$$

d) Since on both sides, the waves have oscillated for integer multiples of $\frac{\pi}{2}$, the probability is simply proportional to the length of the region and the amplitude squared. Therefore $P_R = 4P_L$; and

$$\boxed{P_L = \frac{1}{5}} ; \boxed{P_R = \frac{4}{5}}$$

II-4

a) The states are

$$|\emptyset\rangle \ , \ |\uparrow\rangle \ , \ |\downarrow\rangle \ , \ |\uparrow\downarrow\rangle$$

Of course, the last state with 2 electrons has Fermionic symmetry

$$|\uparrow\downarrow\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}$$

b)

$$\boxed{\mathcal{Z}(\beta, \mu) = 1 + 2e^{-\beta E + \beta\mu} + e^{-\beta(2E - V - 2\mu)}}$$

$$\langle n \rangle = \frac{1}{\beta \mathcal{Z}} \frac{\partial \mathcal{Z}}{\partial \mu} = \boxed{\frac{2e^{-\beta(E-\mu)} + 2e^{-\beta(2E-V-2\mu)}}{1 + 2e^{-\beta(E-\mu)} + e^{-\beta(2E-V-2\mu)}}}$$

This is equal to 1 if and only if

$$\boxed{2E - V - 2\mu = 0}$$

c) Defining

$$\Delta \equiv \frac{geB\hbar}{4m}$$

the magnetic susceptibility becomes

$$\alpha = \frac{\Delta}{B^2} \frac{2e^{-\beta(E-\mu)} \sinh(\beta\Delta)}{1 + 2e^{-\beta(E-\mu)} \cosh(\beta\Delta) + e^{-\beta(2E-V-2\mu)}}$$

for small external field and $\langle n \rangle = 1$, this becomes

$$\boxed{\alpha = \beta(\frac{ge\hbar}{4m})^2 \frac{e^{-\beta V/2}}{1 + e^{-\beta V/2}}}$$

For large βV, single electron states are not very probable, the site is either empty or occupied by 2 electrons in a singlet state; either way, the probable states do not respond effectively to an external magnetic field hence $\alpha \to 0$. On the other hand, in the limit $\beta V \ll 1$, all 4 states are almost equally probable; in other words, half the sites respond to the external field and half do not. This explains the 1/2 thermodynamic factor.