PSU Physics PhD Qualifying Exam Solutions
Fall 2008

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June 15, 2022

Candidacy Exam (with correction)
Department of Physics
August 23, 2008
Part I

Instructions:

- The following problems are intended to probe your understanding of basic physical principles. When answering each question, indicate the principles being applied and any approximations required to arrive at your solution. *If information you need is not given, you may define a variable or make a reasonable physical estimate, as appropriate.* Your solutions will be evaluated based on clarity of physical reasoning, clarity of presentation, and accuracy.
- Please use a new blue book for each question. Remember to write your name and the problem number of the cover of each book.
- We suggest you read all *four* of the problems before beginning to work them. You should reserve time to attempt every problem.

Fundamental constants, conversions, etc.:

Avogadro's number	N_A	$6.022 \times 10^{23}\,\mathrm{mol}^{-1}$
Boltzmann's constant	k_B	$1.381 \times 10^{-23}\,\mathrm{J\,K}^{-1}$
Electron charge magnitude	e	$1.602 \times 10^{-19}\,\mathrm{C}$
Gas constant	R	$8.314\,\mathrm{J\,mol}^{-1}\,\mathrm{K}^{-1}$
Planck's constant	h $\hbar = h/2\pi$	$6.626 \times 10^{-34}\,\mathrm{J\,s}$ $1.055 \times 10^{-34}\,\mathrm{J\,s}$
Speed of light in vacuum	c	$2.998 \times 10^{8}\,\mathrm{m\,s}^{-1}$
Permittivity constant	ϵ_0	$8.854 \times 10^{-12}\,\mathrm{F\,m}^{-1}$
Permeability constant	μ_0	$1.257 \times 10^{-6}\,\mathrm{N\,A}^{-2}$
Gravitational constant	G	$6.674 \times 10^{-11}\,\mathrm{m}^3\,\mathrm{kg}^{-1}\,\mathrm{s}^{-2}$
Standard atmospheric pressure	1 atmosphere	$1.01 \times 10^{5}\,\mathrm{N\,m}^{-2}$
Stefan-Boltzmann constant	σ	$5.67 \times 10^{-8}\,\mathrm{W\,m}^{-2}\,\mathrm{K}^{-4}$
Electron rest mass	m_e	$9.109 \times 10^{-31}\,\mathrm{kg} = 0.5110\,\mathrm{MeV}\,c^{-2}$
Proton rest mass	m_p	$1.673 \times 10^{-27}\,\mathrm{kg} = 938.3\,\mathrm{MeV}\,c^{-2}$
Origin of temperature scales		$0\,^\circ\mathrm{C} = 273\,\mathrm{K}$
1 large calorie (as in nutrition) 1 inch		$4.184\,\mathrm{kJ}$ $2.54\,\mathrm{cm}$

Definite integrals:

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}. \tag{I–1}$$

$$\int_0^{\infty} x^n e^{-x} dx = \Gamma(n+1) = n!. \tag{I–2}$$

Indefinite integrals:

$$\int \frac{1}{\sqrt{x^2+a^2}} dx = \ln\left(x + \sqrt{x^2+a^2}\right). \tag{I–3}$$

$$\int \frac{1}{x^2+a^2} dx = \frac{1}{a} \arctan \frac{x}{a}. \tag{I–4}$$

$$\int \frac{1}{(x^2+a^2)^{3/2}} dx = \frac{x}{a^2\sqrt{x^2+a^2}}. \tag{I–5}$$

I–1. A magnetic field of magnitude B points in the direction $\hat{z}$. The quantum-mechanical Hamiltonian for the spin states of an electron in this field is $H = \omega S_z$, where $\omega = \frac{|e|B}{m_e C}$, and S_z is the projection of the electron's spin on the z-axis.

(a) Write down the time evolution operator which relates the states of the system at time $t > 0$ to the states at time $t = 0$ (in the Schrödinger picture).

(b) What are the stationary states of this system?

(c) If the initial state $|\alpha, t_0 = 0\rangle$ is an eigenstate of the S_x operator (the spin projection along the $\hat{x}$ axis) with eigenvalue $\frac{1}{2}\hbar$ what is the state $|\alpha, t > 0\rangle$

(d) Evaluate $\langle \alpha, t|S_x|\alpha, t\rangle$, $\langle \alpha, t|S_y|\alpha, t\rangle$ and $\langle \alpha, t|S_z|\alpha, t\rangle$ and argue that the spin precesses around the $\hat{z}$ axis.

I–2. (a) Consider a non-relativistic particle of mass m in a spherically symmetric classical potential energy $V(r)$, where r is the distance of the particle from a fixed point. Derive an equation for the radial motion of the form

$$m\frac{d^2r}{dt^2} = -\frac{dV_{\text{eff}}(r; L)}{dr}, \tag{I–6}$$

where the V_{eff} is an effective potential for radial motion, and depends on the angular momentum L.

(b) In general relativity, it is found that an equation of this form still applies to the motion of a small object around another heavy spherical object of mass M. But the effective potential energy is now

$$V_{\text{eff}}(r; L) = \frac{m}{2}\left(1 - \frac{2GM}{c^2 r}\right)\left(\frac{L^2}{m^2 r^2} + c^2\right). \tag{I–7}$$

Show that in the limit $c \to \infty$, this is equivalent to the situation for Newtonian gravity.

The remaining parts of this problem refer to motion governed by Eq. (I–7).

(c) For which values of L do circular orbits exist when the effective potential energy is (I–7)? Find the radii of stable circular orbits and their angular velocities ω_ϕ.

(d) If an orbiting object is slightly disturbed from a stable circular orbit at radius R, it oscillates around the stable radius. Compute the corresponding oscillation frequency ω_r. Expand the precession rate $\omega_p := \omega_\phi - \omega_r$ for $R \gg GM/c^2$ to leading non-vanishing order.

I–3. A hollow, empty, irregularly shaped conductor is charged to a potential V relative to infinity. Someone has calculated the electrostatic potential ϕ for the system, and far from the conductor the result is

$$\phi(\mathbf{r}) = \frac{AV}{r} + \frac{BVz}{r^3} + \frac{CV}{r^3}\left(3\frac{z^2}{r^2} - 1\right) \tag{I–8}$$

where r^2 is $|\mathbf{r}|^2 = x^2 + y^2 + z^2$, and A, B, and C are constants.

(a) What is leading-order contribution to the *electric field* at large distances from the conductor?

(b) What is the charge on the conductor?

(c) What is the capacitance of the conductor?

I–4. A monatomic classical gas of molecular weight M has a temperature T at low pressure. There are N molecules in a volume V. The velocity distribution function for the velocity of the molecules is

$$f(v) = K \exp[-\beta(v_x^2 + v_y^2 + v_z^2)], \tag{I–9}$$

where K is a constant. (Thus the number of molecules in a volume element $d^3\mathbf{v}$ is $f(v)\, d^3\mathbf{v}$.)

(a) What is the value of β?

(b) Derive an equation for the distribution of the speed of the molecules, i.e. a function $g(v)$ such that $g(v)dv$ is the probability of finding a particle with speed v to $v + dv$.

(c) Now let a small hole of area A be made in a container of such a gas, to give an effusive molecular beam. Obtain an analytic formula for the rate at which molecules leave the container. Explain why the distribution of molecular speeds emerging from the hole has a different shape to that you found in the previous part.

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II–1. A copper wire of diameter $d = 1\,\mathrm{mm}$ is shaped in the form of a vertical square loop with sides of length $a = 10\,\mathrm{cm}$. It falls with velocity v from a region where there is a constant horizontal magnetic field of magnitude 1.2 T perpendicular to the plane of the loop, into a region with zero magnetic field, as shown in the figure below. The resistivity of the wire is $\rho_e = 1.7 \times 10^{-8}\,\Omega \cdot \mathrm{m}$, and its density is $\rho_m = 8960\,\mathrm{kg/m^3}$.

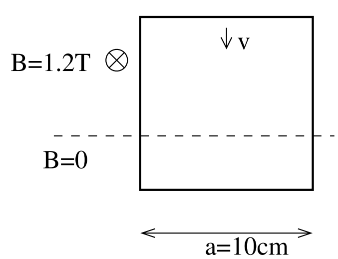


Figure II–1: For problem II–1.

(a) In what direction does a current flow around this lop, and what is its magnitude in terms of the velocity of fall, v?

(b) Show that this gives an upward force on the loop.

(c) Find an expression for the terminal velocity. (I.e., the velocity at which the acceleration would be zero, while the boundary between the two regions of field intersects the circuit.) Estimate the value of the terminal velocity, taking $g = 9.8\,\mathrm{m/s^2}$.

II–2. In this problem you will investigate the conditions limiting the braking ability of a bicycle.

The bicycle moves on a horizontal road in a straight line. See Fig. II–2 for the geometry. The distance between the points of contact of the wheels is 1.05 m. The center of gravity of the bicycle-plus-rider system is 1.15 m above the ground and 0.40 m in front of the point of contact of the rear wheel. In the following calculations, ignore the rotational inertia of the wheels.

There is a coefficient of friction $\mu = 0.8$ between each of the tires and the road.

(a) When only the brake on the rear wheel is applied, what is the maximum deceleration? What causes the limit?

(b) Repeat when only the front brake is used.

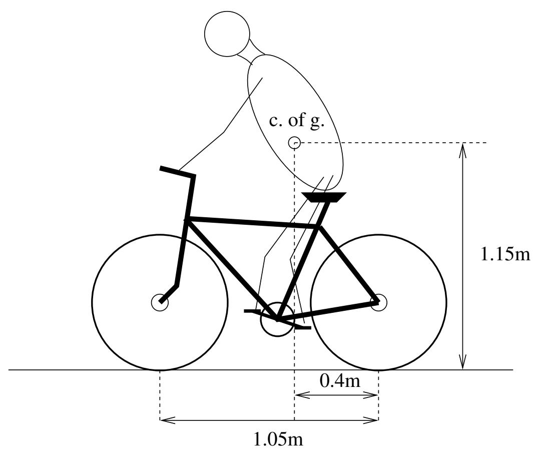


Figure II–2: Geometry of bicycle, for problem II–2.

II–3. In the variational method for estimating the solution to a quantum-mechanical bound-state problem, show that a variational estimate for the ground-state energy is equal to or larger than the true ground-state energy.

Apply the variational method to a particle of mass m in a spherically symmetric Yukawa potential

$$V(r) = \frac{g^2 e^{-r\mu}}{r}, \tag{II–6}$$

where r is the distance between the particle and the fixed center of force. Use a trial wave function of the form

$$\psi(r) \propto e^{-r/a}. \tag{II–7}$$

Here a is a parameter to be determined. It will be sufficient to carry through your calculation to where a and the ground state energy can be determined by an algebraic calculation.

II–4. Consider an ideal gas (of 1 mole) that obeys the usual equation $PV = RT$. The internal energy depends on T, and is independent of P and V.

(a) Starting from the first law of thermodynamics, show that

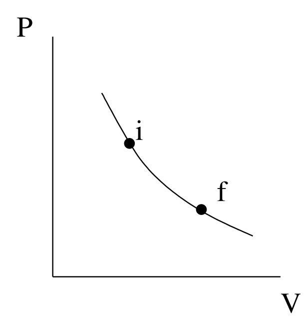
$$C_P - C_V = R. \tag{II–8}$$

(b) Show that for a quasistatic adiabatic process

$$PV^\gamma = \text{constant}, \tag{II–9}$$

where $\gamma = C_P/C_V$. (Assume the specific heats are constant.)

(c) A sketch of the P-V diagram of an isothermal expansion from an initial state i to a final state f is shown below. Sketch the curve of an adiabatic process starting from the *same initial* state. Does the temperature rise or fall in the process?



I-1

a)

$$U(t) = e^{-i\omega t S_z} = e^{-i\frac{\omega t}{2}}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$
$$= \boxed{\begin{pmatrix}e^{-i\omega t/2} & \\ & e^{+i\omega t/2}\end{pmatrix}}$$

b) These are the up and down spin states

$$\boxed{|\uparrow\rangle \equiv \begin{pmatrix}1\\0\end{pmatrix} \ ; \ |\downarrow\rangle = \begin{pmatrix}0\\1\end{pmatrix}}$$

c)

$$|\alpha, t\rangle = U(t)\,|\alpha, 0\rangle = \begin{pmatrix}e^{-i\omega t/2} & \\ & e^{+i\omega t/2}\end{pmatrix}\begin{pmatrix}1/\sqrt{2}\\ 1/\sqrt{2}\end{pmatrix}$$
$$= \boxed{\frac{1}{\sqrt{2}}\begin{pmatrix}e^{-i\omega t/2}\\ e^{+i\omega t/2}\end{pmatrix}}$$

d)

$$\langle\alpha, t|\, S_x\,|\alpha, t\rangle = \frac{1}{2}(1 \ \ 1)\begin{pmatrix}e^{+i\omega t/2} & \\ & e^{-i\omega t/2}\end{pmatrix}\begin{pmatrix} & 1/2\\ 1/2 & \end{pmatrix}\begin{pmatrix}e^{-i\omega t/2} & \\ & e^{+i\omega t/2}\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} = \boxed{\frac{1}{2}\cos\omega t}$$
$$\langle\alpha, t|\, S_y\,|\alpha, t\rangle = \frac{1}{2}(1 \ \ 1)\begin{pmatrix}e^{+i\omega t/2} & \\ & e^{-i\omega t/2}\end{pmatrix}\begin{pmatrix} & -i/2\\ +i/2 & \end{pmatrix}\begin{pmatrix}e^{-i\omega t/2} & \\ & e^{+i\omega t/2}\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} = \boxed{\frac{1}{2}\sin\omega t}$$
$$\langle\alpha, t|\, S_z\,|\alpha, t\rangle = \langle\alpha, 0|\, e^{+i\omega t S_z} S_z e^{-i\omega t S_z}\,|\alpha, 0\rangle = \langle\alpha, 0|\, S_z\,|\alpha, 0\rangle = \boxed{0}$$

In summary

$$\langle\alpha, t|\,\mathbf{S}\,|\alpha, t\rangle = \frac{1}{2}(\hat{\mathbf{x}}\cos\omega t \ + \ \hat{\mathbf{y}}\sin\omega t)$$

which describes a rotating vector in the $x - y$ plane.

I-2

In this problem, the system of units is chosen in a way that

$$c = 4\pi G = \frac{M}{2\pi} = 1$$

we also define

$$l \equiv \frac{L}{m} \ ; \ u \equiv \frac{V}{m}$$

a) Starting from the Lagrangian

$$\frac{1}{2}m\big[\dot{r}^2 \ + r^2(\dot{\theta}^2 + \sin^2\theta\,\dot{\varphi}^2)\big] - V(r)$$

the Euler-Lagrange equation for r coordinate gives

$$
\begin{aligned}
m\ddot{r} &= mr(\dot{\theta}^2 + \sin^2\theta\,\dot{\varphi}^2) - V'(r) \\
&= mr\frac{v_\perp^2}{r^2} - V'(r) = m\frac{l^2}{r^3} - V'(r)
\end{aligned}
$$

which is equivalent to

$$
\boxed{\ddot{r} = -\frac{d}{dr}\left[u(r) + \frac{l^2}{2r^2}\right]}
$$

b) Let us first simplify the potential by reducing a constant term of $\frac{m}{2}$. This is

$$
\begin{aligned}
u_{\text{eff.}}(r) &= \frac{1}{2}(1 - 1/r)(1 + l^2/r^2) \\
&\equiv \frac{-GM}{r} + \frac{l^2}{2r^2} - \frac{GMl^2}{c^2r^3}
\end{aligned}
$$

clearly, in the limit $c^2 \to \infty$, the last term vanishes and we are back to Newtonian gravity.

c) We need a stationary point $u'(r) = 0$

$$
\begin{aligned}
u' &= \frac{1}{2}\left(\frac{1}{r^2} - \frac{2l^2}{r^3} + \frac{3l^2}{r^4}\right) \\
&= \frac{1}{2r^4}\left[(r - l^2)^2 - l^2(l^2 - 3)\right]
\end{aligned}
$$

This means that circular orbits exist if and only if

$$
\boxed{l \geq \sqrt{3}}
$$

Checking the sign of the expression for u', we see that the stable orbit is located at

$$
\boxed{R = l^2 + l\sqrt{l^2 - 3}}
$$

the angular velocity is

$$
\boxed{\omega_\varphi = \frac{l}{R^2} = \frac{1}{l\left(l + \sqrt{l^2 - 3}\right)^2}}
$$

d) The classical limit is equivalent to $l \gg 1$. Up to second order in $1/l$, we have

$$
\begin{aligned}
R &\approx 2l^2\left(1 - \frac{3}{4l^2}\right) \\
\omega_\varphi &\approx \frac{1}{4l^3}\left(1 + \frac{3}{2l^2}\right) \\
\omega_r = \sqrt{u''(R)} &= \sqrt{\frac{-1}{R^3} + \frac{3l^2}{R^4} - \frac{6l^2}{R^5}} \\
\approx \left[\frac{-1}{8l^6}\left(1 + \frac{9}{4l^2}\right) + \frac{3}{16l^6}\left(1 + \frac{3}{l^2}\right) - \frac{6}{32l^8}\right]^{1/2} \\
&\approx \frac{1}{4l^3}\left(1 + \frac{3}{4l^2}\right)
\end{aligned}
$$

Therefore the leading order term in ω_p is

$$
\begin{aligned}
\omega_p &= \frac{3}{16l^5} + \cdots \\
&= \boxed{\frac{3G^4M^4}{c^2l^5} + \cdots}
\end{aligned}
$$

I-3

a) This will be the monopole term

$$\boxed{\mathbf{E}_{\text{mono.}} = \frac{AV\hat{\mathbf{r}}}{r^2}}$$

b) Using the Gauss law on a very large spherical surface, the charge is found to be

$$\boxed{Q = 4\pi\varepsilon_0 AV}$$

c) This is

$$C = \frac{Q}{V} = \boxed{4\pi\varepsilon_0 A}$$

I-4

a)

$$\beta v^2 = \frac{Mv^2/2}{k_B T}$$

leading to

$$\boxed{\beta = \frac{M}{2k_B T}}$$

b) The probability differential is given by

$$dP = 4\pi v^2 dv \times K \exp\big(-\beta v^2\big)$$

therefore

$$\boxed{g(v) = 4\pi K\, v^2 \exp\big(-\beta v^2\big)}$$

c) Let's set up a coordinate system with z axis perpendicular to the hole area and pointing inside the container. The number of particles in a small volume element with speed between v and $v + dv$ is

$$\frac{N}{V} g(v) r^2 \sin\theta \, dv \, dr \, d\theta \, d\varphi$$

The probability that one of these is headed towards the hole is then

$$\frac{A\cos\theta}{4\pi r^2}$$

Such particles will leave the container before some time δt if

$$r < v\delta t$$

Therefore the net number of particles leaving the container before some time t is

$$\begin{aligned}\delta N &= \int_0^\infty dv \int_0^{v\delta t} dr \int_0^{\pi/2} d\theta \int_0^{2\pi} d\varphi \, \frac{N}{V} g(v) \frac{A\cos\theta}{4\pi} \sin\theta \\ &= \frac{AN\langle v\rangle \delta t}{4V}\end{aligned}$$

To find the average speed, first we normalise the distribution as

$$K = \left(\frac{\beta}{\pi}\right)^{3/2}$$

then find that

$$\langle v \rangle = \sqrt{\frac{8k_BT}{\pi M}}$$

and finally

$$\boxed{\dot{N} = \frac{NA}{V}\sqrt{\frac{k_BT}{2\pi M}}}$$

The particles which move with greater speed are more likely to exit the hole in a given time span, this means that the distribution of the speed of the exiting particles will be different.

II-1

a) The resistivity of the loop is

$$R = \rho_e \frac{4a}{\pi d^2/4} = \frac{16\rho_e a}{\pi d^2}$$

The induced emf is trying to keep the magnetic flux constant and therefore **the current will run clock wise.** The magnitude of the emf is given by the rate of the change of the magnetic flux

$$\mathcal{E} = vaB$$

therefore

$$\boxed{I = \frac{\pi d^2 vB}{16\rho_e} \approx 13.9\, A.s/m \ \times v}$$

b) It does ($\mathbf{F} = I\,\mathbf{l} \times \mathbf{B}$); the magnitude is

$$\boxed{F = \frac{\pi d^2 a B^2 v}{16\rho_e}}$$

c) The upward force should equal the weight

$$\rho_m g \times 4a \times \frac{\pi d^2}{4} = \frac{\pi d^2 a B^2 v}{16\rho_e}$$

$$\boxed{v = \frac{16\rho_e\rho_m g}{B^2} \approx 1.66\, cm/s}$$

II-2

The CM height will be denoted by h; its horizontal distance from the rear wheel by x, and the distance between wheels by L. The normal forces N_R, and N_F will respectively denote the normal force between the ground and the rear/front wheels.

We don't want any of the wheels to rise above the ground and this gives us the bound on the brake acceleration.

a) The rear wheel friction is (at most) $N_R\mu$, the other wheel, having no moment of inertia, can not bear friction. The equations of motion are

$$\begin{cases} ma = N_R\mu \\ \\ N_R + N_F = mg \\ \\ N_R x + N_R\mu h = N_F(L-x) \end{cases}$$

Solved as

$$N_R = mg\frac{L-x}{L+\mu h}$$
$$N_F = mg\frac{x+\mu h}{L+\mu h}$$
$$a = \mu g\frac{L-x}{L+\mu h} \approx \boxed{0.26\,g}$$

Although the rear normal force is reduced, it does not become negative no matter how large μ is. The acceleration is only limited by the friction constant μ and the geometry of the bicycle.

b) This time, the equations are

$$\begin{cases} ma = N_F\mu \\ \\ N_R + N_F = mg \\ \\ N_R x + N_F\mu h = N_F(L-x) \end{cases}$$

Solved as

$$N_R = mg\frac{L-x-\mu h}{L-\mu h}$$
$$N_F = mg\frac{x}{L-\mu h}$$
$$a = \mu g\frac{x}{L-\mu h}$$

The fact that N_R is negative with the given value of μ means that this time, the deceleration is not only limited by the friction constant. In fact, if the rider applies a large brake force, then he will face a painful pitchover! The best he can do is to use the brakes to impose a partial slip on the front wheel, equivalent to some $\mu_{\text{eff.}} < 0.8$ such that $\mu_{\text{eff.}}h = L - x$. The deceleration is, then

$$a_F = g\frac{L-x}{h} \approx \boxed{0.57\,g}$$

II-3

For any function, $|\psi\rangle$, we have

$$\langle\psi|\,H\,|\psi\rangle - E_1 = \sum_n |\psi_n|^2(E_n - E_1) \geq 0$$

which is what we wanted to prove.

The trial wave-function is

$$\psi = \frac{\exp(-r/2a)}{\sqrt{8\pi a^3}}$$

with expected values

$$V(a) = \frac{-g^2}{8\pi a^3}\int_0^\infty e^{-r/a}\,\frac{e^{-\mu r}}{r}\,4\pi r^2 dr \;=\; \frac{-g^2}{2a(1+a\mu)}$$

$$T(a) = \frac{-1}{16\pi a^3 m}\int_0^\infty 4\pi r^2 dr\, e^{-r/a}\,\frac{1}{r^2}\Big(-\frac{r}{a}+\frac{r^2}{4a^2}\Big) = \frac{1}{8ma^2}$$

The net energy is therefore

$$E(a) = \frac{1}{8ma^2} - \frac{g^2}{2a(1+a\mu)}$$

The minimum occurs at

$$\frac{dE}{da} = 0 \iff (1+a\mu)^2 = 2mg^2 a(1+2a\mu)$$

solved as

$$\mu a^* = \frac{\sqrt{1+4\alpha-\alpha^2}-1}{4-\alpha}$$

where

$$\alpha \equiv \frac{\mu}{mg^2}$$

The ground state energy upper bound is

$$\boxed{E^* = \frac{\mu g^2}{2}(4-\alpha)^2\Big[\frac{\alpha}{4(\sqrt{1+4\alpha-\alpha^2}-1)^2} - \frac{1}{(\sqrt{1+4\alpha-\alpha^2}-1)(\sqrt{1+4\alpha-\alpha^2}+3-\alpha)}\Big]}$$

II-4

a) For a process with constant volume

$$đQ_V = C_V dT = dU$$

For an isobaric process, then

$$đQ_P = C_P dT = dU - đW_P = dU + PdV = C_V dT + PdV$$

therefore in an isobaric process

$$(C_P - C_V)d\Big(\frac{PV}{R}\Big) = PdV$$

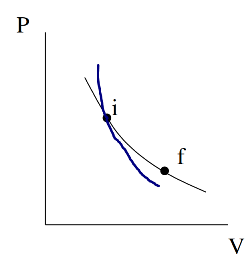
which leads to

$$\boxed{C_P - C_V = R}$$

b)

$$dU = C_V dT = C_V d\Big(\frac{PV}{R}\Big) = đW = -PdV \iff \frac{dP}{P} + (1+R/C_V)\frac{dV}{V} = 0 \iff \boxed{PV^\gamma = \text{constant.}}$$

c)



As the volume increases, the temperature decreases.