PSU Physics PhD Qualifying Exam Solutions
Spring 2008

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Candidacy Exam
Department of Physics
January 19, 2008
Part I

Instructions:

- The following problems are intended to probe your understanding of basic physical principles. When answering each question, indicate the principles being applied and any approximations required to arrive at your solution. *If information you need is not given, you may define a variable or make a reasonable physical estimate, as appropriate.* Your solutions will be evaluated based on clarity of physical reasoning, clarity of presentation, and accuracy.
- Please use a new blue book for each question. Remember to write your name and the problem number of the cover of each book.
- We suggest you read all *four* of the problems before beginning to work them. You should reserve time to attempt every problem.

Fundamental constants, conversions, etc.:

Avogadro's number	N_A	$6.022 \times 10^{23}\,\mathrm{mol}^{-1}$
Boltzmann's constant	k_B	$1.381 \times 10^{-23}\,\mathrm{J\,K}^{-1}$
Electron charge magnitude	e	$1.602 \times 10^{-19}\,\mathrm{C}$
Gas constant	R	$8.314\,\mathrm{J\,mol}^{-1}\,\mathrm{K}^{-1}$
Planck's constant	h $\hbar = h/2\pi$	$6.626 \times 10^{-34}\,\mathrm{J\,s}$ $1.055 \times 10^{-34}\,\mathrm{J\,s}$
Speed of light in vacuum	c	$2.998 \times 10^{8}\,\mathrm{m\,s}^{-1}$
Permittivity constant	ϵ_0	$8.854 \times 10^{-12}\,\mathrm{F\,m}^{-1}$
Permeability constant	μ_0	$1.257 \times 10^{-6}\,\mathrm{N\,A}^{-2}$
Gravitational constant	G	$6.674 \times 10^{-11}\,\mathrm{m}^3\,\mathrm{kg}^{-1}\,\mathrm{s}^{-2}$
Standard atmospheric pressure	1 atmosphere	$1.01 \times 10^{5}\,\mathrm{N\,m}^{-2}$
Stefan-Boltzmann constant	σ	$5.67 \times 10^{-8}\,\mathrm{W\,m}^{-2}\,\mathrm{K}^{-4}$
Electron rest mass	m_e	$9.109 \times 10^{-31}\,\mathrm{kg} = 0.5110\,\mathrm{MeV}\,c^{-2}$
Proton rest mass	m_p	$1.673 \times 10^{-27}\,\mathrm{kg} = 938.3\,\mathrm{MeV}\,c^{-2}$
Origin of temperature scales		$0\,^\circ\mathrm{C} = 273\,\mathrm{K}$
1 large calorie (as in nutrition) 1 inch		4.184 kJ 2.54 cm

Definite integrals:

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}. \tag{I–1}$$

$$\int_{0}^{\infty} x^n e^{-x} dx = \Gamma(n+1) = n!. \tag{I–2}$$

Indefinite integrals:

$$\int \frac{1}{\sqrt{x^2+a^2}} dx = \ln\left(x + \sqrt{x^2+a^2}\right). \tag{I–3}$$

$$\int \frac{1}{x^2+a^2} dx = \frac{1}{a} \arctan \frac{x}{a}. \tag{I–4}$$

$$\int \frac{1}{(x^2+a^2)^{3/2}} dx = \frac{x}{a^2\sqrt{x^2+a^2}}. \tag{I–5}$$

I–1. Consider two objects of mass m and M connected by a spring of spring constant k. They are restricted to motion along a straight line. Determine the frequency of oscillation of the two objects.

I–2. (a) A non-relativistic quantum particle of mass m exists on a line in an infinite square well potential between $x = 0$ and $x = a$. It has an initial wave function $\psi(x) = 4(5a)^{-\frac{1}{2}} \sin^3(\pi x/a)$. Show that this is a linear combination $\psi(x) = c_1\psi_1(x) + c_3\psi_3(x)$ of the ground state ψ_1 and the second excited state ψ_3. Compute the constants c_1 and c_3.

[Hint: You might find it helpful to derive $\sin(3y) = 3\sin y - 4\sin^3 y$, e.g., using $e^{3iy} = (e^{iy})^3$.]

(b) Find the time dependent wave function $\Psi(x, t)$ which at time $t = 0$ is given by $\Psi(x, 0) = \psi(x)$ as defined in part (a), and the probability density to find the particle at position x.

I–3. Obtain the general solution for the classical motion of an electron in free space, in the presence of *both* (i) a constant magnetic field B in the $\hat{\mathbf{z}}$ direction, and (ii) a constant electric field E in the $\hat{\mathbf{x}}$ direction. Describe the motion.

I–4. For a quantum mechanical oscillator with mass m oscillating in one dimension with angular frequency ω:

(a) Calculate the partition function for a single oscillator

(b) Find the internal energy, entropy and the heat capacity of a system consisting of N such oscillators as a function of temperature. Assume the interaction energy between the oscillators can be neglected.

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II–1. A small air bubble of initial radius $r_1 = 1.0\,\mathrm{cm}$ is introduced at the bottom of a column of water 10 m high. What is the diameter of the bubble after it rises to the surface?

II–2. A tritium atom (hydrogen of mass number three) in its ground state undergoes beta decay to a $^3\mathrm{He}^+$ ion. Assume that the only important consequence is a sudden change of a neutron into a proton. Use this approximation to find the probabilities:

(a) That the electron remains in its ground state.

(b) That the electron is excited to the $n = 2$ excited state.

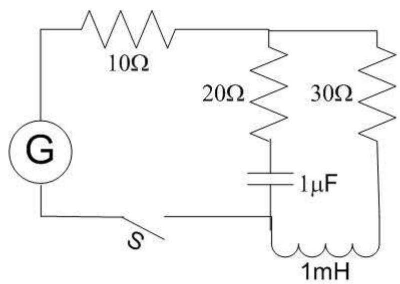
You will need to use one or more of the following radial functions for hydrogen-like atoms, of nuclear charge Z. (You do not need to derive the wave functions, just to use them correctly.)

$$R_{1s}(\rho) = 2Z^{3/2} e^{-Z\rho} \tag{II–6}$$

$$R_{2s}(\rho) = \left(\frac{Z}{2}\right)^{3/2} (2 - Z\rho) e^{-Z\rho/2} \tag{II–7}$$

$$R_{3s}(\rho) = 2\left(\frac{Z}{3}\right)^{3/2} \left[1 - \frac{2}{3}Z\rho + \frac{2}{27}(Z\rho)^2\right] e^{-Z\rho/3}. \tag{II–8}$$

Here $\rho = r/a_0$, where a_0 is the Bohr radius.



II–3.

(a) Assume the power source "G" in the circuit shown above provides 10 V DC. The capacitor is initially uncharged.

  i. Find the current through the 10 Ω resistor a very short time after switch "S" is closed.

  ii. Find the current through the 10 Ω resistor a very long time after switch "S" is closed.

(b) Now assume the power source G provides 10 V AC, at variable frequency. The capacitor is initially uncharged. The switch is closed. Find the power dissipated in the 10 Ω resistor when the frequency is set to a very small value. Repeat this calculation for when the frequency is set to a very large value.

II–4. A bar has a cross sectional area A, length L, and thermal conductivity K. One end is connected to a thermal reservoir at temperature T_c (cold), and the other to T_h (hot). What is the rate of entropy generation dS/dt? The sides of the bar are thermally insulated.

I-1

Sitting on one of the masses, the other has an effective mass of

$$\mu \equiv \frac{mM}{m+M}$$

aka the reduced mass. Therefore the frequency is

$$\boxed{\omega = \sqrt{\frac{k(m+M)}{mM}}}$$

I-2

a)

$$\sin^3\theta = \sin^2\theta \times \sin\theta = \frac{1-\cos 2\theta}{2}\sin\theta = \frac{1}{2}\sin\theta - \frac{1}{2}\sin\theta\cos 2\theta$$
$$= \boxed{\frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta}$$

Now it's easy to write the given wave function as a linear combination

$$\psi = \frac{4}{\sqrt{5a}}\sin^3\frac{\pi x}{a} = \frac{1}{\sqrt{5a}}\big[3\sin\frac{\pi x}{a} - \sin\frac{3\pi x}{a}\big]$$
$$= \boxed{\frac{3}{\sqrt{10}}\psi_1 - \frac{1}{\sqrt{10}}\psi_3}$$

b)

$$\psi(x,t) = \frac{3}{\sqrt{10}}\psi_1(x)\,e^{-\frac{i\pi^2 t}{2ma^2}} - \frac{1}{\sqrt{10}}\psi_3(x)e^{-\frac{9i\pi^2 t}{2ma^2}}$$
$$= \frac{1}{\sqrt{10}}e^{-\frac{i\pi^2 t}{2ma^2}}\sqrt{\frac{2}{a}}\big[3\sin\frac{\pi x}{a} - \sin\frac{3\pi x}{a}\,e^{-\frac{4i\pi^2 t}{ma^2}}\big]$$

The probability density is given by the squared absolute value

$$\boxed{\rho(x,t) = \frac{1}{5a}\big[9\sin^2\frac{\pi x}{a} + \sin^2\frac{3\pi x}{a} - 6\sin\frac{\pi x}{a}\sin\frac{3\pi x}{a}\cos\frac{4\pi^2 t}{ma^2}\big]}$$

I-3

The equations of motion are

$$m\frac{d\mathbf{v}}{dt} = -e\big(\mathbf{E} + \mathbf{v}\times\mathbf{B}\big)$$

or in components

$$\begin{cases} \ddot{x} = -\frac{e}{m}(E + B\dot{y}) \\ \ddot{y} = \frac{eB}{m}\dot{x} \\ \ddot{z} = 0 \end{cases}$$

The last equation is solved as

$$\boxed{z(t) = z(0) + t\dot{z}(0)}$$

Integrating the first equation, we get

$$\dot{x} = \dot{x}(0) - \frac{eE}{m}t - \frac{eB}{m}y + \frac{eB}{m}y(0)$$

plugged in the second equation, this implies

$$\ddot{y} + (\frac{eB}{m})^2 y = \frac{eB}{m}\left[\dot{x}(0) - \frac{eE}{m}t + \frac{eB}{m}y(0)\right]$$

solved as

$$\boxed{y(t) = y(0) + \frac{m}{eB}\dot{x}(0)(1 - \cos\frac{eBt}{m}) - \frac{Et}{B} + \frac{mE}{eB^2}\sin\frac{eBt}{m}}$$

This can in turn be used to find

$$\boxed{x(t) = x(0) + \frac{m\dot{x}(0)}{eB}\sin\frac{eBt}{m} - \frac{mE}{eB^2}(1 - \cos\frac{eBt}{m})}$$

I-4

a)

$$Z = \sum_{n=0}^{\infty} e^{-\beta\omega(n+1/2)} = \frac{e^{-\beta\omega/2}}{1 - e^{-\beta\omega}} = \boxed{\frac{1}{2\sinh(\beta\omega/2)}}$$

b)

$$Z = \left[2\sinh(\beta\omega/2)\right]^{-N}$$

and therefore

$$\langle E\rangle = -\partial_\beta \log Z = \boxed{\frac{N\omega}{2}\coth(\beta\omega/2)}$$

$$C = \frac{\partial\langle E\rangle}{\partial T} = -k_B\beta^2\frac{\partial\langle E\rangle}{\partial\beta^2} = \boxed{Nk_B\left[\frac{\beta\omega/2}{\sinh(\beta\omega/2)}\right]^2}$$

$$S = \frac{E-F}{T} = k_B\beta(E + \frac{1}{\beta}\log Z)$$

$$= \boxed{Nk_B\left\{\frac{\beta\omega}{2}\coth\left(\frac{\beta\omega}{2}\right) - \log\left[2\sinh\left(\frac{\beta\omega}{2}\right)\right]\right\}}$$

II-1

Assuming constant temperature and an atmospheric pressure of about $10m$ of water, the ideal gas equation of state $pV = nRT$ implies that the volume of the bubble will double. This means

$$\boxed{d_2 \approx 2.52\,cm}$$

II-2

a) Neglecting the difference between the reduced masses and the mass of the electron ($m_n \approx m_p$), the probability is given as

$$P = |\langle 100|1'0'0'\rangle|^2 = \left|\int_0^\infty dr\, r^2 (2e^{-r})(2\times 2\sqrt{2}e^{-2r})\right|^2$$

$$= 128\left|\int_0^\infty x^2 e^{-3x} dx\right|^2 = \boxed{\frac{512}{729} \approx 0.702}$$

b)

$$P_2 = |\langle 100|2'0'0'\rangle|^2 = \left|\int_0^\infty dr\, r^2 (2e^{-r})(2-2r)e^{-r}\right|^2 = 16\left|\int_0^\infty dx\, x^2(1-x)e^{-2x}\right|^2$$

$$16\left|\frac{2}{8} - \frac{6}{16}\right| = \boxed{0.25}$$

II-3

a)

i - A short time after the switch is closed, the capacitor is equivalent to a short circuit and an inductor is equivalent to an open circuit. The circuit is then easily solved to reveal the current as

$$I = \frac{10\,V}{10\,\Omega + 20\,\Omega} = \boxed{\frac{1}{3}\,A}$$

ii - This time, the inductor is short and the capacitor is open

$$I = \frac{10\,V}{10\,\Omega + 30\,\Omega} = \boxed{0.25\,A}$$

b) In an AC circuit, a capacitor has impedance

$$Z = \frac{1}{j\omega C}$$

and an inductor

$$Z = j\omega L$$

this means that in small frequencies, the capacitor is open and the inductor is short. Therefore the current through the resistor is $0.25\,A$s. Assuming that these are the r.m.s. vlues, the power is

$$P = RI^2 = \boxed{0.625\,W}$$

For large frequencies, the capacitor is short and the inductor is open.

$$P = 10\,\Omega\,\times(1/3\ A)^2 = \boxed{\frac{10}{9}\,W \approx 1.11\,W}$$

II-4

If some head $đQ$ is transferred from a system with temperature $T + dT$ to another at temperature T, the net change in entropy is

$$dS = \frac{đQ dT}{T^2}$$

and the entropy increase rate is

$$\frac{dS}{dt} = \dot{Q} \frac{dT}{T^2}$$

Summed along the bar, this is

$$\left(\frac{dS}{dt}\right)_{\text{total}} = \dot{Q} \int \frac{dT}{T^2} = \boxed{\frac{KA}{L}\left(1 - \frac{T_c}{T_h}\right)^2}$$