# PSU Physics PhD Qualifying Exam Solutions
# Fall 2009

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July 3, 2022

# Candidacy Exam (corrected)
# Department of Physics
# October 3, 2009
## Part I

Instructions:

- The following problems are intended to probe your understanding of basic physical principles. When answering each question, indicate the principles being applied and any approximations required to arrive at your solution. *If information you need is not given, you may define a variable or make a reasonable physical estimate, as appropriate.* Your solutions will be evaluated based on clarity of physical reasoning, clarity of presentation, and accuracy.
- Please use a new blue book for each question. Remember to write your name and the problem number of the cover of each book.
- We suggest you read all *four* of the problems before beginning to work them. You should reserve time to attempt every problem.

Fundamental constants, conversions, etc.:

| | | |
|---|---|---|
| Avogadro's number | $N_A$ | $6.022 \times 10^{23}\,\mathrm{mol}^{-1}$ |
| Boltzmann's constant | $k_B$ | $1.381 \times 10^{-23}\,\mathrm{J\,K}^{-1}$ |
| Electron charge magnitude | $e$ | $1.602 \times 10^{-19}\,\mathrm{C}$ |
| Gas constant | $R$ | $8.314\,\mathrm{J\,mol}^{-1}\,\mathrm{K}^{-1}$ |
| Planck's constant | $h$ | $6.626 \times 10^{-34}\,\mathrm{J\,s}$ |
| | $\hbar = h/2\pi$ | $1.055 \times 10^{-34}\,\mathrm{J\,s}$ |
| Speed of light in vacuum | $c$ | $2.998 \times 10^{8}\,\mathrm{m\,s}^{-1}$ |
| Permittivity constant | $\epsilon_0$ | $8.854 \times 10^{-12}\,\mathrm{F\,m}^{-1}$ |
| Permeability constant | $\mu_0$ | $1.257 \times 10^{-6}\,\mathrm{N\,A}^{-2}$ |
| Gravitational constant | $G$ | $6.674 \times 10^{-11}\,\mathrm{m}^3\,\mathrm{kg}^{-1}\,\mathrm{s}^{-2}$ |
| Standard atmospheric pressure | 1 atmosphere | $1.01 \times 10^{5}\,\mathrm{N\,m}^{-2}$ |
| Stefan-Boltzmann constant | $\sigma$ | $5.67 \times 10^{-8}\,\mathrm{W\,m}^{-2}\,\mathrm{K}^{-4}$ |
| Electron rest mass | $m_e$ | $9.109 \times 10^{-31}\,\mathrm{kg} = 0.5110\,\mathrm{MeV}\,c^{-2}$ |
| Proton rest mass | $m_p$ | $1.673 \times 10^{-27}\,\mathrm{kg} = 938.3\,\mathrm{MeV}\,c^{-2}$ |
| Origin of temperature scales | | $0\,^\circ\mathrm{C} = 273\,\mathrm{K}$ |
| 1 large calorie (as in nutrition) | | $4.184\,\mathrm{kJ}$ |
| 1 inch | | $2.54\,\mathrm{cm}$ |

I–1. A uniform spherical solid ball rolls with a center-of-mass velocity $v$. The ball is of mass $m$ and radius $a$.

(a) What is its total kinetic energy?

(b) The ball rolls on parabolic track of the form $y(x) = \beta x^2$, where $\beta$ is a positive constant, $x$ is horizontal position, and $y$ is vertical position. As the ball rolls back and forth, the maximum height of its center of mass is $h_0$. What is its maximum velocity?

I–2. A quantum mechanical particle of mass $m$ is situated on a line with the following potential:

$$V(x) = \begin{cases} 0 & : \quad 0 < x < a, \\ +\infty & : \quad x < 0,\ x > a. \end{cases} \tag{I–1}$$

Its initial wave function is $\psi(x, t = 0) = A \sin^3(\pi x/a)$. Find the wave function at arbitrary time $t > 0$. Does the particle return to the initial state at some time $T$?

I–3. Consider a rectangular wave guide along the $z$-axis, assuming the electric and magnetic fields to be of the form

$$\vec{H} = \mathrm{Re}\left[\vec{h}(x, y) e^{i(\omega t - k_z z)}\right], \quad \vec{E} = \mathrm{Re}\left[\vec{e}(x, y) e^{i(\omega t - k_z z)}\right]. \tag{I–2}$$

(a) Use the $x$- and $y$-components of Faraday's and Ampère's laws to determine the transversal field components in terms of $e_z$ and $h_z$.

(b) Use Maxwell's equations to derive the Helmholtz equation

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + p^2 f = 0 \tag{I–3}$$

for $e_z$ and $h_z$, and identify $p$.

(c) Solve the Helmholtz equations for TM modes, where $H_z = 0$, by separation of variables. Which frequencies $\omega$ are possible for given side lengths $a$ and $b$ of the wave guide's cross-section.

I–4. An atom has a magnetic moment $\vec{\mu}$. In a particular crystal it can only be oriented in any of the six directions $\pm x$, or $\pm y$, or $\pm z$, and in the absence of a magnetic field, each of these 6 states has the same energy.

Now assume that a fixed magnetic field points in the $\hat{z}$-direction.

(a) What is the partition function at nonzero temperatures for the magnetic system?

(b) Derive a formula for the average magnetic moment $\langle \vec{\mu} \rangle$ as a function of temperature and magnetic field.

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II–1. A newtonian particle of mass $m$ falls under the influence of the earth's gravity (treated as uniform). It experiences a viscous drag force $kv$, where $k$ is a constant and $v$ is the particle's speed. The particle released from rest at a height $h_0$ at time $t = 0$. Obtain its height $h(t)$ as a function of time $t$?

II–2. Two identical quantum-mechanical particles of mass $m$ are confined to a line where there is a potential $V(x)$. All the *single-particle* energy eigenstates are bound states and are non-degenerate. Let the single-particle energy eigenvalues be $E_1$, $E_2$, etc, in order of increasing energy. You should neglect all inter-particle interactions.

(a) For the *two-particle system*, what are the two lowest eigenenergies when the particles are spinless bosons? Express your answer in terms of the single-particle energies.

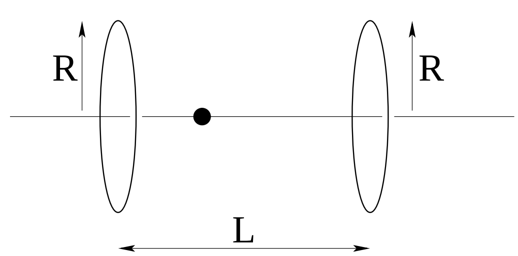
(b) What if they are spinless fermions?

(c) What if they are spin-1/2 fermions?

II–3. Two rings of radii $R$ are placed in parallel planes at distance $L$ from each other such that the line connecting their centers is orthogonal on the planes (see figure). Each ring has a uniform positive charge density and total charge $Q$. A small bead of positive charge $q$ and mass $m$ is constrained to move on the line connecting the centers of the two rings. Find:

(a) The equilibrium position of the bead.

(b) The period of small oscillations about that position.



II–4. An enclosure contains a classical ideal gas at pressure $\overline{p}$ and has in one of its walls a small hole of area $A$ through which molecules pass into vacuum. In this vacuum, directly in front of the hole at a distance $L$ from it, there is a suspended circular disk of radius $R$. It is oriented such that the normal to its surface points towards the hole. Assuming that the molecules escaping from the enclosure are scattered elastically from the disk, find the force exerted on the disk by the escaping molecules, in the limit that the size of the hole is much less than $L$ and $R$. Base your answer on the Maxwell-Boltzmann distribution.

# I-1

a) The angular velocity is

$$\omega = \frac{v}{a}$$

threfore the energy is

$$T = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}mv^2 + \frac{1}{2}\frac{2}{5}ma^2\frac{v^2}{a^2} = \boxed{\frac{7}{10}mv^2}$$

b) Let $\xi$ denote the $x-$coordinate value of the touching point between the sphere and the curve. Then the position of the center of mass is given by

$$x = \xi - a\sin[\theta(\xi)] \quad ; \quad y = f(\xi) + a\cos[\theta(\xi)]$$

where $\theta \equiv \arctan[f'(\xi)]$ is the angle the curve makes with the horizontal. A geometric argument, proves the rolling angular velocity as

$$a\omega = \frac{ds}{dt} - a\frac{d\theta}{dt} = \dot{\xi}\left[\sec\theta - a\frac{d\theta}{d\xi}\right]$$

The kinetic energy comprises of two parts: a translational and a rotational one. The translational one is

$$T_t = \frac{1}{2}m\dot{\xi}^2\Big[1 + a^2\theta'^2\cos^2\theta - 2a\theta'\cos\theta + \tan^2\theta + a^2\theta'^2\sin^2\theta - 2a\theta'\tan\theta\sin\theta\Big]$$

$$= \frac{1}{2}m\dot{\xi}^2\Big(\frac{1}{\cos\theta} - a\theta'\Big)^2$$

The rotational one is

$$T_r = \frac{1}{5}ma^2\omega^2$$

$$= \frac{1}{5}m\dot{\xi}^2\Big(\frac{1}{\cos\theta} - a\theta'\Big)^2$$

The fact that $T_r$ is proportional to $T_t$ leads to

$$v_{CM} = \sqrt{\frac{10T}{7m}}$$

Attaining its maximum when all the potential energy is converted to kinetic energy

$$\boxed{v^*_{CM} = \sqrt{\frac{10}{7}g(h_0 - a)}}$$

# I-2

$$\psi = A\sin^3\left(\frac{\pi x}{a}\right) = A\sin\left(\frac{\pi x}{a}\right)\frac{1-\cos(2\pi x/a)}{2}$$

$$= \frac{A}{2}\left[\sin\left(\frac{\pi x}{a}\right) - \frac{1}{2}\sin\left(\frac{3\pi x}{a}\right) + \frac{1}{2}\sin\left(\frac{\pi x}{a}\right)\right]$$

$$= \frac{A}{4}\left[3\sin\left(\frac{\pi x}{a}\right) - \sin\left(\frac{3\pi x}{a}\right)\right]$$

$$= \frac{3}{\sqrt{10}}\psi_1(x) - \frac{1}{\sqrt{10}}\psi_3(x)$$

Evolving throught time, this becomes

$$\boxed{\psi(x,t) = \frac{\exp\left(-i\pi^2 t/2ma^2\right)}{\sqrt{5a}}\left[3\sin\left(\frac{\pi x}{a}\right) - \exp\left(-4i\pi^2 t/ma^2\right)\sin\left(\frac{3\pi x}{a}\right)\right]}$$

After some time

$$T_1 = \frac{ma^2}{2\pi}$$

the state will be *equivalent* to its initial state; but it takes

$$T_2 = \frac{4ma^2}{\pi}$$

for the state to fully recover back to its initial state.

# I-3

a) These components of Maxwell's equations read

$$\frac{\partial e_z}{\partial y} - ik_z e_y = i\omega\mu h_x \qquad ik_z e_x - \frac{\partial e_z}{\partial x} = i\omega\mu h_y$$

$$\frac{\partial h_z}{\partial y} - ik_z h_y = -i\omega\varepsilon e_x \qquad ik_z h_x - \frac{\partial h_z}{\partial x} = -i\omega\varepsilon e_z$$

and are solved as

$$\boxed{\mathbf{e}_2 = \frac{i}{\omega^2\varepsilon\mu - k_z^2}\Big(k_z \boldsymbol{\nabla}_2 e_z + \omega\mu\hat{\mathbf{z}} \times \boldsymbol{\nabla}_2 h_z\Big)}$$

$$\boxed{\mathbf{h}_2 = \frac{i}{\omega^2\varepsilon\mu - k_z^2}\Big(k_z \boldsymbol{\nabla}_2 h_z - \omega\varepsilon\hat{\mathbf{z}} \times \boldsymbol{\nabla}_2 e_z\Big)}$$

b)

$$\boldsymbol{\nabla}_2 . \mathbf{h}_2 + ik_z h_z = 0$$

is equivalent to

$$\boxed{\Big[\nabla_2^2 + (\omega^2\varepsilon\mu - k_z^2)\Big] h_z = 0}$$

the same equation is similarly found to govern $e_z$.

c) For the $h_z = 0$ modes, all the boundary conditions are summarized in $e_z = 0$ at the boundaries. Then the Helmholtz equation is solved as

$$e_z^{(mn)} = A \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right)$$

with

$$\omega_{mn}^2 = \frac{k_z^2}{\varepsilon\mu} + \frac{\pi^2}{\varepsilon\mu}\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right) > \boxed{\frac{\pi^2}{\varepsilon\mu}(1/a^2 + 1/b^2)}$$

# I-4

a) Let $x \equiv \beta B \mu$

$$\boxed{Z = e^x + e^{-x} + 4}$$

b)

$$\langle \boldsymbol{\mu} \rangle = \mu\hat{\mathbf{z}} \frac{e^{+x} - e^{-x}}{e^{+x} + e^{-x} + 4}$$

$$= \boxed{\mu\hat{\mathbf{z}}\frac{\sinh(x)}{2+\cosh(x)}}$$

## II-1

Let us start by finding the velocity

$$\frac{dv}{dt} = g - \frac{k}{m}v$$

Adding the initial condition $v(0) = 0$, this equation is solved as

$$v(t) = \frac{mg}{k}\left(1 - e^{-kt/m}\right)$$

The dropping distance is the integral of this velocity over time

$$x(t) = \frac{mg}{k}\left[t - \frac{m}{k}\left(1 - e^{-kt/m}\right)\right]$$

Leading to the height function

$$\boxed{h(t) = h_0 - \frac{mg}{k}\left[t - \frac{m}{k}\left(1 - e^{-kt/m}\right)\right]}$$

## II-2

a) $E_1 + E_1\ ,\ E_1 + E_2$

b) $E_1 + E_2\ ,\ E_1 + E_3$

c) $E_1 + E_1\ ,\ E_1 + E_2$

## II-3

a) Let $\xi$ be the displacement of the charge from the center. Also define

$$a \equiv \frac{L}{2R} \quad ; \quad x \equiv \frac{\xi}{R}$$

then the potential energy is

$$U = \frac{qQ}{4\pi\varepsilon_0 R}\left\{\left[1 + (a - x)^2\right]^{-1/2} + \left[1 + (a + x)^2\right]^{-1/2}\right\}$$

The equilibrium condition is $dU/dx = 0$ equivalent to

$$\frac{B_+}{(1 + B_+)^3} = \frac{B_-}{(1 + B_-)^3}$$

where

$$B_\pm \equiv (a \pm x)^2$$

In the process of getting to the equation above, two irrelevent answers with $|x| > a$ have been added. The equation may be further simplified into

$$(B_+ - B_-)\Big[1 - 3B_+B_- - B_+B_-(B_+ + B_-)\Big] = 0$$

the obvious solution is

$$B_+ - B_- = 0 \iff x = 0$$

or the center. The two other solutions (if existing) are found by solving the (essentially third order) equation

$$1 = (a^2 - x^2)^2 \Big[3 + 2(a^2 + x^2)\Big]$$

It turns out that at $a^2 = 1/2$, the stable equilibrium in the center turns into an unstable equilibrium in the center surrounded by two symmetrically posed stable equilibriums.

b) To find the oscillations frequency, (only in the center), we expand the potential to second order to find the quadratic term to be

$$U_{\text{quad.}} = \frac{1}{2} m \frac{qQ(L^2 - 2R^2)}{8\pi\varepsilon_0 m(R^2 + L^2/4)^{5/2}} \xi^2$$

leading to

$$\boxed{\omega^2 = \frac{qQ(L^2 - 2R^2)}{8\pi\varepsilon_0 m(R^2 + L^2/4)^{5/2}}}$$

## II-4

The force is the sum of bits

$$dF = 2mv_z \frac{\delta N}{\delta t}$$

where $\delta N$ is the number of particles that exit the hole in the time interval $\delta t$, are aimed at the disk and have $z-$velocitis between $v_z$ and $v_z + dv_z$. Let $f$ denote the probability density function for $v_z$ and $g$ for the absolute value of the $x - y$ plane velocity. Then

$$\begin{aligned}
\delta N &= \int_0^{v_z \delta t} dz \int_0^{Rz/L} 2\pi s ds\, n f(v_z) dv_z\, g(sv_z/z) \frac{Av_z}{z} \frac{1}{2\pi s} \\
&= An\delta t f(v_z) v_z dv_z \mathbb{P}[\sqrt{v_x^2 + v_y^2} \leq \frac{Rv_z}{L}]
\end{aligned}$$

To find the net force, we need to integrate this over $v_z$. Defining

$$\sigma^2 \equiv \frac{k_B T}{m}$$

we have

$$\begin{aligned}
F &= 2mnA \int_0^\infty v_z^2 f(v_z) dv_z \mathbb{P}[\sqrt{v_x^2 + v_y^2} \leq \frac{Rv_z}{L}] \\
&= 2mnA \int_0^\infty v_z^2 f(v_z) dv_z \Big(1 - e^{-R^2 v_z^2 / 2\sigma^2 L^2}\Big) \\
&= mnA\sigma^2 \Big[1 - \Big(\frac{L^2}{L^2 + R^2}\Big)^{3/2}\Big] \\
&= \boxed{pA\Big[1 - \Big(\frac{L^2}{L^2 + R^2}\Big)^{3/2}\Big]}
\end{aligned}$$