PSU Physics PhD Qualifying Exam Solutions
Spring 2009

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Candidacy Exam
Department of Physics
January 17, 2009
Part I

Instructions:

- The following problems are intended to probe your understanding of basic physical principles. When answering each question, indicate the principles being applied and any approximations required to arrive at your solution. *If information you need is not given, you may define a variable or make a reasonable physical estimate, as appropriate.* Your solutions will be evaluated based on clarity of physical reasoning, clarity of presentation, and accuracy.
- Please use a new blue book for each question. Remember to write your name and the problem number of the cover of each book.
- We suggest you read all *four* of the problems before beginning to work them. You should reserve time to attempt every problem.

Fundamental constants, conversions, etc.:

Avogadro's number	N_A	$6.022 \times 10^{23}\,\mathrm{mol^{-1}}$
Boltzmann's constant	k_B	$1.381 \times 10^{-23}\,\mathrm{J\,K^{-1}}$
Electron charge magnitude	e	$1.602 \times 10^{-19}\,\mathrm{C}$
Gas constant	R	$8.314\,\mathrm{J\,mol^{-1}\,K^{-1}}$
Planck's constant	h	$6.626 \times 10^{-34}\,\mathrm{J\,s}$
	$\hbar = h/2\pi$	$1.055 \times 10^{-34}\,\mathrm{J\,s}$
Speed of light in vacuum	c	$2.998 \times 10^{8}\,\mathrm{m\,s^{-1}}$
Permittivity constant	ϵ_0	$8.854 \times 10^{-12}\,\mathrm{F\,m^{-1}}$
Permeability constant	μ_0	$1.257 \times 10^{-6}\,\mathrm{N\,A^{-2}}$
Gravitational constant	G	$6.674 \times 10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}}$
Standard atmospheric pressure	1 atmosphere	$1.01 \times 10^{5}\,\mathrm{N\,m^{-2}}$
Stefan-Boltzmann constant	σ	$5.67 \times 10^{-8}\,\mathrm{W\,m^{-2}\,K^{-4}}$
Electron rest mass	m_e	$9.109 \times 10^{-31}\,\mathrm{kg} = 0.5110\,\mathrm{MeV}\,c^{-2}$
Proton rest mass	m_p	$1.673 \times 10^{-27}\,\mathrm{kg} = 938.3\,\mathrm{MeV}\,c^{-2}$
Origin of temperature scales		$0\,^\circ\mathrm{C} = 273\,\mathrm{K}$
1 large calorie (as in nutrition)		$4.184\,\mathrm{kJ}$
1 inch		$2.54\,\mathrm{cm}$

I–1. Consider a spacecraft (mass m) in orbit about Earth (mass $M_\odot \gg m$).

(a) Recall that for an elliptical orbit of eccentricity e and the semi-major axis a, the distances from the center of the orbit are $a(1-e)$ and $a(1+e)$ at the perigee and apogee. (Apogee is the point of furthest excursion from Earth; perigee is the point of closest approach to Earth.)

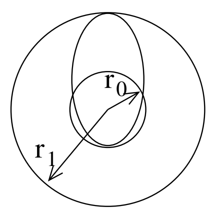
What is the energy when the the spacecraft has speed v and is at a distance r from Earth?

Using conservation of energy and angular momentum, show that

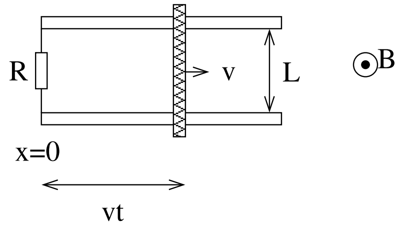
$$v^2 = GM_\odot \left(\frac{2}{r} - \frac{1}{a} \right). \tag{I–1}$$

What are the spacecraft's speeds at the perigee and apogee?

(b) Now consider a set of orbital maneuvers that will move the spacecraft from a circular orbit of radius r_0 to another circular orbit of radius $r_1 > r_0$. We treat the spacecraft's engine as if it provides an impulsive force that will instantaneously change the spacecraft's velocity in a specified direction by a specified amount Δv. What is the Δv_1 required to change the initial circular orbit to an elliptical one with distance r_1 at apogee and r_0 at perigee? What is the Δv_2 required to convert this elliptical orbit into a circular one at distance r_1? When should each of these Δv's be applied, and in what directions?



I–2. Two metal rails, separated by distance L, run parallel to the x-axis, and at $x = 0$, a resistor R is connected across the rails. A closed circuit is formed by a frictionless metal rod which slides along the rails with a constant velocity v such that its position at time t is $x = vt$. (See figure.)



Suppose a constant magnetic field exists perpendicular to the plane of the rails. Neglecting the resistance of the rails and the rod, and any self-inductance in this circuit:

(a) Calculate any current induced in the circuit.

(b) Calculate the external force required to maintain a steady motion of the rod.

(c) Calculate the power required, P_1, to maintain this constant speed.

(d) Compare this power P_1 with that dissipated in the resistor P_2. Explain the result.

I–3. The electron in a hydrogen atom occupies the combined spin and position state

$$\psi = \sqrt{\frac{2}{3}} R_{31}(r) Y_{10}(\theta, \phi)\chi_+ + \sqrt{\frac{1}{3}} R_{43}(r) Y_{32}(\theta, \phi)\chi_- \tag{I–2}$$

Here $R_{nl}(r)$ represents the radial part of the wave function, $Y_{lm}(\theta, \phi)$ are spherical harmonics describing the angular part of the wave function and $\chi_\pm$ are spin wave functions obeying

$$S_z \chi_\pm = \pm\frac{\hbar}{2}\chi_\pm \quad . \tag{I–3}$$

The products $R_{nl}(r)Y_{lm}(\theta, \phi)\chi_+$ and $R_{nl}(r)Y_{lm}(\theta, \phi)\chi_+$ have unit norm.

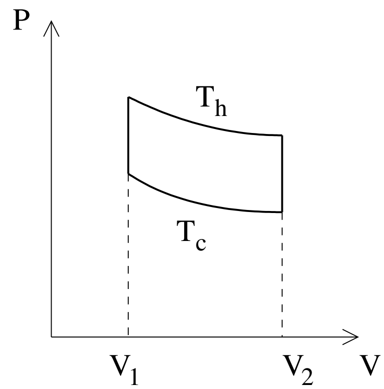
Find the following expectation values:

- $\langle H \rangle$ (in terms of E_0, the binding energy of the ground state).
- $\langle L_z \rangle$
- $\langle S^2 \rangle$
- $\langle S_x \rangle$

I–4. A Stirling cycle for an ideal gas is similar to a Carnot cycle except that the two adiabatic legs are replaced by two constant volume legs (see figure). Two legs are at constant volume V_1, V_2 and and the other two are at constant temperatures T_h, T_c.

(a) Derive the change in: (i) internal energy ΔU, (ii) heat Q, and (iii) work W, for each of four legs of the cycle.

(b) What is the efficiency for converting heat from the hot source to work?



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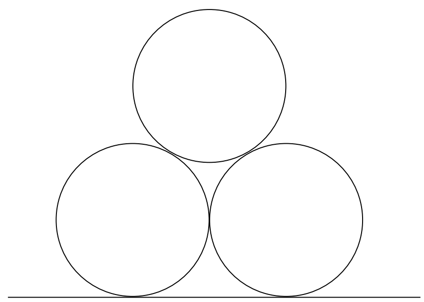
II–1. Consider three identical cylinders, each of them having the same mass m and radius r. One cylinder is supported by the other two in a close packed triangular configuration on a flat surface.

(a) Assume that the lower cylinders can only roll without slipping on the flat surface. What is the minimum coefficient of static friction between cylinders for which the stack would be stable?

(b) If the flat-surface–cylinder coefficient of friction is the same as that of the cylinder–cylinder contact point, which friction surface would slip first?



II–2. (a) Use Maxwell's equations to show that electric charge is conserved by deriving the local conservation law

$$\frac{\partial \rho}{\partial t} + \mathrm{div}\mathbf{j} = 0 \qquad \text{(II–1)}$$

for the charge and current density.

(b) What do Maxwell's equations say about the conservation of magnetic charge?

(c) Also energy must be conserved. Show that the electromagnetic energy density $\mathcal{E} = \frac{1}{2}(\mathbf{E} \cdot \mathbf{D} + \mathbf{B} \cdot \mathbf{H})$ and the energy flux $\mathbf{P} = \mathbf{E} \times \mathbf{H}$, together with a work term of moving charges, satisfy an equation of the form (II–1).

II–3. Consider a quantum square well for a non-relativistic particle of mass m, with a 1D potential:

$$V(x) = \begin{cases} V_0 & : |x| > L/2, \\ 0 & : |x| < L/2. \end{cases} \qquad \text{(II–2)}$$

In the following restrict your attention to wave functions that are even under $x \mapsto -x$.

(a) Show that the energy levels of bound states must satisfy the relation

$$\tan\frac{kL}{2} = \frac{\alpha}{k}, \tag{II–3}$$

where

$$k \equiv \frac{\sqrt{2mE}}{\hbar}, \qquad \alpha \equiv \frac{\sqrt{2m(V_0 - E)}}{\hbar}. \tag{II–4}$$

(b) Determine an expression for the ground state energy in the limit of large V_0.

(c) Determine a non-zero approximate expression for the ground state *binding* energy in the limit of small V_0.

II–4. The entropy per unit particle, $s = S/N$ of a system is given in terms of the energy per particle $u = U/N$ and the volume per particle $v = V/N$ by

$$s = s_0 + R\ln\frac{v-b}{v_0-b} + \frac{3}{2}R\,g(u+a/v) \quad . \tag{II–5}$$

for some function g.

(a) Derive the equation of state $f(P, v, T) = 0$.

(b) In the case that $g\,(u + a/v) = \ln\sinh(c(u + a/v))$, show that

$$C_V = \frac{3}{2}\frac{R}{1-\left(\frac{3}{2}RcT\right)^2}. \tag{II–6}$$

I-1

a)

$$E = -\frac{GM_\odot m}{r} + \frac{1}{2}mv^2$$

Gives

$$v^2 = \frac{2E}{m} + \frac{2GM_\odot}{r}$$

It remains to show

$$E = -\frac{GM_\odot m}{2a}$$

To see this, note that the conservation laws are

$$v_A a(1+e) = v_P a(1-e)$$

$$\frac{1}{2}v_A^2 - \frac{GM_\odot}{a(1+e)} = \frac{1}{2}v_P^2 - \frac{GM_\odot}{a(1-e)}$$

leading to

$$v_A^2\Big[\frac{(1+e)^2-(1-e)^2}{(1-e)^2}\Big] = \frac{GM_\odot}{a}\Big(\frac{1}{1-e} - \frac{1}{1+e}\Big) \implies v_A^2 = \frac{GM_\odot}{a}\frac{1-e}{1+e}$$

substituting v_A^2 in the energy formula at the apogee, we get

$$E = -\frac{GM_\odot m}{2a}$$

which is what we wanted to show. ■

b) Using the result from the previous part, we may write

$$v_0^2 = GM_\odot\Big(\frac{2}{r_0} - \frac{1}{r_0}\Big) =$$

$$v_P^2 = GM_\odot\Big(\frac{2}{r_0} - \frac{2}{r_0+r_1}\Big) = \frac{2GM_\odot r_1}{r_0(r_0+r_1)}$$

$$\boxed{\Delta v_1 = \sqrt{\frac{GM_\odot}{r_0}}\Big(\sqrt{\frac{2r_1}{r_0+r_1}} - 1\Big)}$$

$$v_A^2 = GM_\odot\Big(\frac{2}{r_1} - \frac{2}{r_0+r_1}\Big) = \frac{2GM_\odot r_0}{r_1(r_0+r_1)}$$

$$v_1^2 = \frac{GM_\odot}{r_1}$$

$$\boxed{\Delta v_2 = \sqrt{\frac{GM_\odot}{r_1}}\Big(1 - \sqrt{\frac{2r_0}{r_0+r_1}}\Big)}$$

I-2

a)

$$\mathcal{E} = BLv \quad \Rightarrow \quad \boxed{I = \frac{BLv}{R} \quad \text{C.W.}}$$

b)

$$\boxed{F = \frac{B^2 L^2 v}{R}}$$

c)

$$\boxed{P_1 = \frac{B^2 L^2 v^2}{R}}$$

d)

$$\boxed{P_2 = RI^2 = \frac{B^2 L^2 v^2}{R}}$$

In accordance with energy conservation.

I-3

$$\langle H \rangle = \frac{2}{3}E_3 + \frac{1}{3}E_4 = -\frac{E_0}{3}(\frac{2}{9} + \frac{1}{16}) = \boxed{-\frac{41E_0}{432}}$$

$$\langle L_z \rangle = \frac{2}{3} \times 0 + \frac{1}{3} \times 2 = \boxed{\frac{2}{3}}$$

$$\langle S^2 \rangle = \frac{1}{2}(\frac{1}{2} + 1) = \boxed{\frac{3}{4}}$$

$$\langle S_x \rangle = \frac{1}{2}(\sqrt{\frac{2}{3}} \quad \sqrt{\frac{1}{3}}) \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \begin{pmatrix} \sqrt{2/3} \\ \sqrt{1/3} \end{pmatrix} = \boxed{\frac{\sqrt{2}}{3}}$$

I-4

a)

	T_h	V_2	T_c	V_1
ΔU	0	$-C_V(T_h - T_c)$	0	$C_V(T_h - T_c)$
Q	$nRT_h \log \frac{V_2}{V_1}$	$-C_V(T_h - T_c)$	$-nRT_c \log \frac{V_2}{V_1}$	$C_V(T_h - T_c)$
W	$-nRT_h \log \frac{V_2}{V_1}$	0	$nRT_c \log \frac{V_2}{V_1}$	0

b)

$$\boxed{\eta = \frac{nR(T_h - T_c)\log(V_2/V_1)}{nRT_h \log(V_2/V_1) + C_V(T_h - T_c)}}$$

II-1

a) Let N be the normal force between the ground surface and each of the bottom cylinders.

$$N = \frac{3}{2}w$$

where w is the weight of each of the cylinders. The contact surface between the two bottom cylinders can have no friction (non-normal) forces due to the symmetry of the problem (which is assumed to be reflected in the forces too). This means that the two friction forces between the ground and the bottom cylinders and the top cylinder-bottom cylinder surface are the same. We call this force f. The ground-cylinder friction force is assumed to move the cylinders towards each other, the direction of the other friction is therefore downwards. The normal force between the top and bottom cylinders is denoted by N' and finally, the normal force between the two bottom cylinders is denoted by $n > 0$. The vertical equilibrium equations read

$$\begin{cases} \sqrt{3}N' + f = w \\ n + \frac{1}{2}N' = (1 + \frac{\sqrt{3}}{2})f \end{cases}$$

which are solved as

$$N' = \frac{(1+\sqrt{3}/2)w - n}{2+\sqrt{3}}$$

$$f = \frac{w/2 + \sqrt{3}n}{2+\sqrt{3}}$$

this is viable if

$$\mu \geq \frac{f}{N'} = \frac{w + 2\sqrt{3}n}{(2+\sqrt{3})w - 2n} \geq \boxed{\frac{1}{2+\sqrt{3}}}$$

b) For the ground surface to not slip, we need

$$\mu \geq \frac{f}{N} = \frac{w + 2\sqrt{3}n}{3w(2+\sqrt{3})} \geq \frac{1}{3(2+\sqrt{3})}$$

therefore, the top cylinder-bottom cylinder surface slips first.

II-2

a) Applying the divergence operator to

$$\boldsymbol{\nabla} \times \mathbf{B} = \mu_0(\mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t})$$

one gets

$$\boldsymbol{\nabla} \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0$$

which is what we wanted to show. ■

b) Most importantly, the $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$ law explicitly says that there are no magnetic charges! Assuming that this law is modified, the $\boldsymbol{\nabla} \times \mathbf{E}$ law gives

$$\frac{\partial \rho_m}{\partial t} = 0$$

c)

$$\frac{\partial \mathcal{E}}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{P} = \frac{1}{2}\Big(\frac{\partial \mathbf{E}}{\partial t}.\mathbf{D} + \mathbf{E}.\frac{\partial \mathbf{D}}{\partial t} + \frac{\partial \mathbf{B}}{\partial t}.\mathbf{H} + \mathbf{B}.\frac{\partial \mathbf{H}}{\partial t}\Big) + \mathbf{H}.\boldsymbol{\nabla} \times \mathbf{E} - \mathbf{E}.\boldsymbol{\nabla} \times \mathbf{H}$$

$$= \frac{1}{2}\Big(\frac{\partial \mathbf{E}}{\partial t}.\mathbf{D} - \mathbf{E}.\frac{\partial \mathbf{D}}{\partial t} - \frac{\partial \mathbf{B}}{\partial t}.\mathbf{H} + \mathbf{B}.\frac{\partial \mathbf{H}}{\partial t}\Big) - \mathbf{E}.\mathbf{J}$$

For a linear system with symmetric (Hermitian) magnetic permeability and electric permittivity, the expression inside the parentheses vanishes and we get

$$\boxed{\frac{\partial \mathcal{E}}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{P} + \mathbf{E}.\mathbf{J} = 0}$$

II-3

a) The wave function has to be of the form

$$\psi(x) = \begin{cases} A\cos kx & |x| \le L/2 \\ e^{-\alpha|x|} & |x| \ge L/2 \end{cases}$$

in order for it to be normalisable (no exponentially growing tail.) Boundary conditions read

$$A\cos(kL/2) = \exp(-\alpha L/2)$$

$$kA\sin(kL/2) = \alpha\exp(-\alpha L/2)$$

A may be cancelled by dividing the two equations by each other and finding

$$k\tan(kL/2) = \alpha \quad \blacksquare$$

From now on, we use a system of units in which $m = L = \hbar = 1$.

b, c) Let us use a change of variables from the unknown k to x as

$$k = \pi(1-x) \quad x \ll 1$$

The equation to solve is

$$\sqrt{\frac{2V_0}{\pi^2(1-x)^2} - 1} = \tan\big[\frac{\pi}{2}(1-x)\big]$$

equivalent to

$$\frac{\pi(1-x)}{\sqrt{2V_0}} = \sin\Big(\frac{\pi x}{2}\Big) \qquad *$$

For large V_0, x is small and we can Taylor expand as

$$\frac{1-x}{\sqrt{2V_0}} = \frac{x}{2} - \frac{\pi^2 x^3}{48} + \cdots$$

this leads to

$$x \approx \sqrt{\frac{2}{V_0}}$$

or

$$\boxed{E_g = \frac{\hbar^2\pi^2}{2mL^2}\Big(1 - \sqrt{\frac{8\hbar^2}{V_0 mL^2}}\Big) + \mathcal{O}(1/V_0)}$$

For large V_0, the equation $*$ is satisfied only if $x = 1 - y$ and $y \ll 1$.

$$\frac{\pi y}{\sqrt{2V_0}} = 1 - \frac{\pi^2 y^2}{8} + \cdots$$

solved as

$$y \approx \frac{\sqrt{2V_0}}{\pi}$$

leading to

$$\boxed{E_g = V_0 + \mathcal{O}(V_0^2)}$$

II-4

a) Given $S(E,V,N)$, one can write

$$dS = \frac{\partial S}{\partial E}dE + \frac{\partial S}{\partial V}dV + \frac{\partial S}{\partial N}dN$$

$$= \frac{\partial S}{\partial E}(TdS - pdV + \mu dN) + \frac{\partial S}{\partial V}dV + \frac{\partial S}{\partial N}dN$$

since (S,V,N) are independent variables, this gives three simultaneous equations

$$T = \frac{1}{\partial S/\partial E}$$

$$p = T\frac{\partial S}{\partial V}$$

$$\mu = -T\frac{\partial S}{\partial N}$$

for the given function, these lead to

$$T = \frac{2}{3k_B g'} \quad p = \frac{-a}{v^2} + \frac{2}{3(v-b)g'}$$

leading to

$$\boxed{(p + a/v^2)(v-b) = k_B T}$$

b) In a constant volume process

$$đQ = dE$$

$$dT = \frac{-2g''}{3Nk_B g'^2}dE$$

therefore

$$C_V = \frac{đQ_V}{dT} = -\frac{3Nk_B}{2}\frac{g'^2}{g''}$$

$$= \frac{3}{2}Nk_B \cosh^2\left[c(u + a/v)\right]$$

$$= \frac{3}{2}Nk_B \frac{1}{1 - \tanh^2\left[c(u + a/v)\right]}$$

this is what we wanted to prove since

$$\frac{3}{2}k_B cT = \tanh\left[c(u + a/v)\right]$$

■