PSU Physics PhD Qualifying Exam Solutions
Fall 2010

Koorosh Sadri

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Candidacy Exam
Department of Physics
October 2, 2010
Part I

Instructions:

- The following problems are intended to probe your understanding of basic physical principles. When answering each question, indicate the principles being applied and any approximations required to arrive at your solution. *If information you need is not given, you may define a variable or make a reasonable physical estimate, as appropriate.* Your solutions will be evaluated based on clarity of physical reasoning, clarity of presentation, and accuracy.
- Please use a new blue book for each question. Remember to write your name and the problem number of the cover of each book.
- We suggest you read all *four* of the problems before beginning to work them. You should reserve time to attempt every problem.

Fundamental constants, conversions, etc.:

Avogadro's number	N_A	$6.022 \times 10^{23}\,\mathrm{mol}^{-1}$
Boltzmann's constant	k_B	$1.381 \times 10^{-23}\,\mathrm{J\,K}^{-1}$
Electron charge magnitude	e	$1.602 \times 10^{-19}\,\mathrm{C}$
Gas constant	R	$8.314\,\mathrm{J\,mol}^{-1}\,\mathrm{K}^{-1}$
Planck's constant	h $\hbar = h/2\pi$	$6.626 \times 10^{-34}\,\mathrm{J\,s}$ $1.055 \times 10^{-34}\,\mathrm{J\,s}$
Speed of light in vacuum	c	$2.998 \times 10^{8}\,\mathrm{m\,s}^{-1}$
Permittivity constant	ϵ_0	$8.854 \times 10^{-12}\,\mathrm{F\,m}^{-1}$
Permeability constant	μ_0	$1.257 \times 10^{-6}\,\mathrm{N\,A}^{-2}$
Gravitational constant	G	$6.674 \times 10^{-11}\,\mathrm{m}^3\,\mathrm{kg}^{-1}\,\mathrm{s}^{-2}$
Standard atmospheric pressure	1 atmosphere	$1.01 \times 10^{5}\,\mathrm{N\,m}^{-2}$
Stefan-Boltzmann constant	σ	$5.67 \times 10^{-8}\,\mathrm{W\,m}^{-2}\,\mathrm{K}^{-4}$
Electron rest mass	m_e	$9.109 \times 10^{-31}\,\mathrm{kg} = 0.5110\,\mathrm{MeV}\,c^{-2}$
Proton rest mass	m_p	$1.673 \times 10^{-27}\,\mathrm{kg} = 938.3\,\mathrm{MeV}\,c^{-2}$
Origin of temperature scales		$0\,^\circ\mathrm{C} = 273\,\mathrm{K}$
1 large calorie (as in nutrition) 1 inch		4.184 kJ 2.54 cm

Definite integrals:

$$\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}. \tag{I–1}$$

$$\int_0^\infty x^n e^{-x} dx = \Gamma(n+1) = n!. \tag{I–2}$$

Laplacian in spherical polar coordinates (r, θ, ϕ):

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 f}{\partial \phi^2}. \tag{I–3}$$

Laplacian in cylindrical coordinates (r, θ, z):

$$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}. \tag{I–4}$$

I–1. (a) A rope of length l is hanging over the edge of a table such that the vertically hanging piece has length $x_0 < l$. When released, the rope starts to slide down. Determine the distance $x(t)$ of the bottom end of the hanging piece to the table top as a function of time, ignoring friction and the resistance of the rope to bending.

(b) Consider the situation in part (i), but now taking into account a friction force which is proportional to the weight of the horizontal piece of the rope lying on the table, with proportionality constant μ. Determine $x(t)$ for initial values $x(0) = x_0$ with $0 < x_0 < l$ and $\dot{x}(0) = v_0 > 0$ at all times with positive $\dot{x}$.

I–2. An infinitely long solenoid cylinder extends from $z = -\infty$ to $z = +\infty$ and is concentric around the z axis. The cylinder radius is r, and it is wrapped with turns of wire that carry current I. The part of the solenoid in the region $z < 0$ has n turns of wire per unit length, while for $z > 0$ there are $2n$ turns per unit length. The magnetic field direction along the z axis is everywhere in the $+z$ direction.

An electron of mass m, charge e, and spin $\mathbf{s} = \frac{\hbar}{2}\hat{\mathbf{z}}$, is initially moving in the $+z$ direction inside the cylinder. For $z \ll 0$ the electron moves along the z axis with velocity $\mathbf{V}_i = v_i\hat{\mathbf{z}}$. Traveling into region $z \gg 0$, the electron velocity changes to $\mathbf{V}_f = v_f\hat{\mathbf{z}}$. (Approximate the electron magnetic moment with $\boldsymbol{\mu} = -\frac{e}{m}\mathbf{s}$.)

(a) Determine an expression for the magnetic field in the interior of each solenoid.

(b) What is the force on an electron located along the axis inside the solenoid, for $z \ll 0$, where the field is uniform?

(c) Assuming no spin rotation and non-relativistic kinematics, determine the final speed v_f for $z \gg 0$ in terms of v_i, r, n, I, m, and e.

I–3. In an example related to black body radiation, assume that the number of photon modes per unit angular frequency ω in an empty cubic box of side L is

$$\frac{dN}{d\omega} = \frac{\omega^2 L^3}{\pi^2 c^3}. \tag{I–5}$$

If each of these modes is treated as a photon mode that can be populated with and integer number of photons, the energies for a mode will be restricted to $E = n\hbar\omega$, where n is a non-negative integer $\{0, 1, 2, 3, \ldots\}$.

(a) Calculate the partition function for the set of modes at one particular angular frequency ω. (Be sure to sum the series.)

(b) Use the partition function to determine the average energy for the modes at angular frequency ω at temperature T.

(c) Determine an expression for the total electromagnetic energy in the box. (Set up the integral over frequency but don't integrate it).

(d) Without actually integrating over frequency, it is clear that the energy density will be proportional to T^α. Find α.

I–4. A quantum mechanical particle of mass m is constrained to move between two concentric impermeable spheres of radii $r = a$ and $r = b$. There is no other potential. Find the ground state energy and the corresponding normalized wave function.

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II–1. Consider two different relativistic proton-proton scattering experiments. In the first experiment a beam of protons is accelerated to energy $2E$ and allowed to collide with a target (e.g., liquid hydrogen) at rest in the laboratory. In the second, two beams of protons are each accelerated to energy E and then collided head-on in the laboratory.

(a) Evaluate the collisional energy in the center-of-mass frame for each experiment. Which experiment allows for more energetic collisions?

(b) What is the relationship between the energy of the beam in the first experiment, and the energy of each of the beams in the second experiment, that is required for the two experiments to probe the same energy scale? The LHC will allow for 14 TeV center-of-mass energy proton-proton collisions: What energy would the beam in a stationary target experiment need to be to probe the same energy scale?

II–2. Consider a grounded conducting sphere of radius a the center of which coincides with the origin of the coordinate system. Place a point charge q at position $\mathbf{y}$ outside it.

(a) Find the value q' and the position $\mathbf{y}'$ of the image charge inside the sphere.

(b) Evaluate the potential ϕ at any point $\mathbf{x}$ outside the sphere.

(c) Evaluate the surface charge density σ induced on the surface of the sphere.

II–3. Consider a collection of identical flat disks, confined to the xy-plane, each rotating freely about an axis through its center, i.e., around the z axis. Let the disks have moment of inertia I. Suppose we now treat this as a two-dimensional gas, the disks colliding with each other and transferring rotational kinetic energy such that the angular velocities ω vary over a range $-\infty$ to $+\infty$.

Determine that the average angular momentum at some equilibrium temperature T has the form

$$\langle I|\omega|\rangle = \sqrt{\frac{2Ik_BT}{\pi}}. \tag{II–5}$$

II–4. A system has five orbital states into which one can put three identical particles. How many different arrangements are there for:

(a) spin-zero bosons?

(b) spin one-half fermions?

(c) classical particles?

I-1

a) The potential energy as a function of the hanging length x is given by

$$U(x) = -m\frac{x}{l}g\frac{x}{2} = -\frac{1}{2}\frac{mg}{l}x^2$$

this leads to an antiharmonic motion

$$\boxed{x(t) = x_0 \cosh\left(\sqrt{g/l}\, t\right)}$$

b) This time, the Newton's equation of motion has an extra term

$$\ddot{x} = \frac{g}{l}x - \mu\frac{g}{l}(l - x)$$

solved (with the given initial conditions) as

$$\boxed{x(t) = \frac{\mu l}{1+\mu} + \left(x_0 - \frac{\mu l}{1+\mu}\right)\cosh\left[\sqrt{\frac{g(1+\mu)}{l}}\, t\right]}$$

I-2

a) Regrading this solenoid as the superposition of an infinite solenoid with n windings per unit length and a half-infinite solenoid with n windings per length, we find the magnetic filed (on the $z-$axis) as

$$\boxed{\mathbf{B} = \hat{\mathbf{z}}\mu_0 nI\left[\frac{3}{2} + \frac{z}{2\sqrt{z^2+r^2}}\right]}$$

b) Both the $q\mathbf{v}\times\mathbf{B}$ and $\boldsymbol{\nabla}(\boldsymbol{\mu}.\mathbf{B})$ forces are zero on the $z-$axis and in the $z \ll 0$ region where the field is uniform..

c) The easiest way would be to use the conservation of energy

$$\frac{1}{2}mv_i^2 - \mu\mu_0 nI = \frac{1}{2}mv_f^2 - 2\mu\mu_0 nI$$

Leading to

$$\boxed{v_f = \sqrt{v_i^2 - \frac{e\hbar\mu_0 nI}{m^2}}}$$

I-3

a)

$$Z = \sum_{n=0}^{\infty} e^{-n\beta\hbar\omega} = \boxed{\frac{1}{1-e^{-\beta\hbar\omega}}}$$

b)

$$E(\omega) = -\frac{\partial}{\partial\beta}\log(Z) = \boxed{\frac{\hbar\omega}{e^{\beta\hbar\omega}-1}}$$

c)

$$U = \int_0^\infty E(\omega) \frac{dN}{d\omega} d\omega = \int_0^\infty \frac{\hbar\omega}{e^{\beta\hbar\omega} - 1} \frac{\omega^2 L^3}{\pi^2 c^3} d\omega$$

d)

$$U = \frac{\hbar L^3}{\pi^2 c^3} (\beta\hbar)^{-4} \int_0^\infty \frac{x^3 dx}{e^x - 1} \propto T^4 \quad \Rightarrow \quad \boxed{\alpha = 4}$$

I-4

In the ground state, there is no orbit angular momentum. The (non-normalized) wavefunction is

$$\psi = \frac{1}{r} \sin\left[\frac{\pi(r-a)}{b-a}\right]$$

with energy

$$\boxed{E_g = \frac{\pi^2}{2m(b-a)^2}}$$

II-1

a) I use the letter T to denote the kinetic energy. Also, note that if you want these experiments to be *classically equivalent*, then you need to accelerate the incident particles in the first experiment up to $4T$ and not $2T$. In any case, in the second experiment

$$\boxed{T_{CM}^{(2)} = T}$$

since the LAB frame is already the CM frame. For the first experiment, if we define

$$\gamma \equiv 1 + \frac{2T}{m} \quad ; \quad \beta \equiv \sqrt{1 - \frac{1}{\gamma^2}}$$

then, the CM velocity, β_0 is found by solving the equation

$$\frac{\beta - \beta_0}{1 - \beta\beta_0} = \beta_0$$

as

$$\beta_0 = \frac{1}{\beta} - \sqrt{\frac{1}{\beta^2} - 1}$$

leading to the CM energy

$$T_{CM}^{(1)} = \frac{m}{\sqrt{1-\beta_0^2}} - m = \boxed{m\left(\sqrt{1 + T/m} - 1\right)}$$

To compare the two, note that

$$\frac{T_{CM}^{(2)}}{T_{CM}^{(1)}} = 1 + \sqrt{1 + T/m} > 1$$

and therefore the second experiment is more energetic.

b) If T and β denote the kinetic energy and the velocity (of each particle) in the second experiment, then, in order for the experiments to be equivalent, we need an incident velocity

$$\beta' = \frac{2\beta}{1+\beta^2} \quad \Rightarrow \quad \gamma' = \frac{1+\beta^2}{1-\beta^2}$$

Therefore

$$T' = m(\gamma' - 1) = \boxed{4T(1+T/2m)}$$

For the LHC, using $m_p \approx 938.3\,\mathrm{MeV}/c^2$,this is

$$T' = 4 \times \ 14\,\mathrm{TeV}\ \times \ \sqrt{1 + 14000000/938.3} \approx 6.8\,\mathrm{PeV}$$

II-2

a)

$$\boxed{q' = -\frac{a}{y}q} \quad ; \quad \boxed{\mathbf{y}' = \frac{a^2}{y^2}\mathbf{y}}$$

b)

$$\boxed{\phi = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{|\mathbf{x}-\mathbf{y}|} - \frac{ay}{|y^2\mathbf{x} - a^2\mathbf{y}|}\right]}$$

c)

$$\sigma = \varepsilon_0 E_r = \boxed{\frac{q}{4\pi}\left\{\frac{a - y\cos\theta}{(a^2+y^2-2ay\cos\theta)^{3/2}} - \frac{a^2}{y^2}\frac{y - a\cos\theta}{\left[a^2 + a^4/y^2 - 2(a^3/y)\cos\theta\right]^{3/2}}\right\}}$$

II-3

$$H_{\mathrm{Rot.}} = \frac{L^2}{2I}$$

The distribution of the angular momentum is thereore

$$f(L) = Ae^{-\beta L^2/2I}$$

which is a ceneterred normal distribution with

$$\sigma^2 = I/\beta$$

therefore

$$\langle |L| \rangle = \sigma \int_{-\infty}^{+\infty} \frac{dx}{\sqrt{2\pi}} e^{-x^2/2}\,|x| = \sigma\sqrt{\frac{2}{\pi}} = \boxed{\sqrt{\frac{2Ik_BT}{\pi}}}$$

II-4

a) They can all fit in a single orbital (5), or fit in a 2+1 combination (5×4), or spread in 3 different orbitals ($5C3$). These add up to $\boxed{35}$ bosonic states.

b) It is almost the same as bosons except that all three of them can not fit in a single orbital. This means there are $\boxed{30}$ states for such fermions.

c) Each particle has 5 different options, therefore

$$\Omega = 5^3 = \boxed{125}$$

states.