PSU Physics PhD Qualifying Exam Solutions
Fall 2011

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Candidacy Exam
Department of Physics
October 1, 2011

Part I

Instructions:

- The following problems are intended to probe your understanding of basic physical principles. When answering each question, indicate the principles being applied and any approximations required to arrive at your solution. *If information you need is not given, you may define a variable or make a reasonable physical estimate, as appropriate.* Your solutions will be evaluated based on clarity of physical reasoning, clarity of presentation, and accuracy.
- Please use a new blue book for each question. Remember to write your name and the problem number of the cover of each book.
- We suggest you read all *four* of the problems before beginning to work them. You should reserve time to attempt every problem.

Fundamental constants, conversions, etc.:

Avogadro's number	N_A	$6.022 \times 10^{23}\,\mathrm{mol^{-1}}$
Boltzmann's constant	k_B	$1.381 \times 10^{-23}\,\mathrm{J\,K^{-1}}$
Electron charge magnitude	e	$1.602 \times 10^{-19}\,\mathrm{C}$
Gas constant	R	$8.314\,\mathrm{J\,mol^{-1}\,K^{-1}}$
Planck's constant	h	$6.626 \times 10^{-34}\,\mathrm{J\,s}$
	$\hbar = h/2\pi$	$1.055 \times 10^{-34}\,\mathrm{J\,s}$
Speed of light in vacuum	c	$2.998 \times 10^{8}\,\mathrm{m\,s^{-1}}$
Permittivity constant	ϵ_0	$8.854 \times 10^{-12}\,\mathrm{F\,m^{-1}}$
Permeability constant	μ_0	$1.257 \times 10^{-6}\,\mathrm{N\,A^{-2}}$
Gravitational constant	G	$6.674 \times 10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}}$
Standard atmospheric pressure	1 atmosphere	$1.01 \times 10^{5}\,\mathrm{N\,m^{-2}}$
Stefan-Boltzmann constant	σ	$5.67 \times 10^{-8}\,\mathrm{W\,m^{-2}\,K^{-4}}$
Electron rest mass	m_e	$9.109 \times 10^{-31}\,\mathrm{kg} = 0.5110\,\mathrm{MeV}\,c^{-2}$
Proton rest mass	m_p	$1.673 \times 10^{-27}\,\mathrm{kg} = 938.3\,\mathrm{MeV}\,c^{-2}$
Origin of temperature scales		$0\,^\circ\mathrm{C} = 273\,\mathrm{K}$
1 large calorie (as in nutrition)		$4.184\,\mathrm{kJ}$
1 inch		$2.54\,\mathrm{cm}$

Definite integrals:

$$\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}. \tag{I–1}$$

$$\int_0^\infty x^n e^{-x} dx = \Gamma(n+1) = n!. \tag{I–2}$$

Laplacian in spherical polar coordinates (r, θ, ϕ):

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 f}{\partial \phi^2}. \tag{I–3}$$

Laplacian in cylindrical coordinates (r, θ, z):

$$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}. \tag{I–4}$$

I–1. (a) In an experiment, a particle A of mass m is at rest on a smooth horizontal table. A particle B of mass bm, where $b > 1$, is projected along the table directly towards A with speed u.

The collision is perfectly elastic. Find an expression for the speed of A after the collision in terms of b and u. What is the maximum possible ratio of the final speed of A to the initial speed of B. All motion is along a single line.

(b) In a second experiment, particles B_1, B_2, $\ldots$, B_n and A are in a line on the table as follows:

Particle B_1 is projected directly towards the other particles, with speed u. The other particles B_2, B_3, $\ldots$, B_n, and A are at initially at rest. The mass of B_i $(i = 1, 2, \ldots, n)$ is $\lambda^{n+1-i}m$, where λ is a number that is greater than 1; the mass of A is m. All collisions are perfectly elastic. Show that, by choosing n sufficiently large, there is no upper limit on the speed at which A can be made to move. In the case $\lambda = 4$, determine the least value of n for which A moves at more than $20u$.

I–2. (a) In quantum mechanics, the wave function Ψ for a single particle defines the probability density $\rho = |\Psi|^2$. Find an expression for the probability current $\vec{J}$ such that a continuity equation results.

(b) Consider a scattering problem in one dimension such that stationary states have the asymptotic form $\psi(x) = Ae^{ikx} + Be^{-ikx}$ to the left of the scattering potential, and $\psi(x) = Fe^{ikx} + Ge^{-ikx}$ to the right. The solution of the Schrödinger equation gives a relation between the coefficients of the form

$$\begin{pmatrix} B \\ F \end{pmatrix} = S \begin{pmatrix} A \\ G \end{pmatrix}$$

where S is a matrix called the S-matrix. Using the results of part (a), show that S is unitary.

I–3. (a) Use Maxwell's equations to show that the electromagnetic field satisfies the continuity equation

$$\frac{\partial \rho}{\partial t} + \mathrm{div}\vec{J} = 0\,.$$

(b) Explain how this equation implies charge conservation.

(c) What is the analogous equation for energy conservation of the electromagnetic field?

I–4. A self-contained machine only inputs two equal steady streams of hot and cold water at temperatures T_1 and T_2. Its only output is a single high-speed jet of water. The heat capacity per unit mass of water, C, may be assumed to be independent of temperature. The machine is in a steady state, and the kinetic energy in the incoming jets is negligible.

(a) What is the speed of the outgoing jet in terms of T_1, T_2 and T, where T is the temperature of the outgoing jet?

(b) What is the maximum possible speed of the jet?

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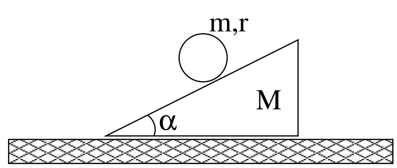
II–1. A hoop of mass m and radius r rolls without slipping down a wedge of mass M. The wedge makes a fixed angle α with the horizontal, and it is on a frictionless horizontal surface:



(a) Derive equations of motion for the system.

(b) Obtain an expression for the position and velocity of the wedge as a function of time.

II–2. (a) An electron of definite momentum starts in a large potential-free region going towards a region where the electrical potential is ϕ. The boundary between the regions is sharp and perpendicular to the momentum of the electron. Find a suitable wave function for this situation.

(b) An experiment to test this is made by aiming a well-collimated beam of electrons in the potential-free region perpendicularly towards the region boundary. The electrical potential in the other region is $\phi = -3\,\text{V}$, the de Broglie wavelength of the electrons is 0.5 nm, and the beam current is $2\,\mu\text{A}$. Determine how much current is reflected at the boundary.

II–3. A perfectly conducting sphere of radius a has no net charge and is not grounded. A constant external electric field $\vec{E} = \hat{z}E$ is applied to the sphere:

(a) What is the electrostatic potential outside of the sphere?

(b) What is the surface charge density on the surface of the sphere?

(c) What is the polarizability of the sphere? (The polarizability α is the ratio of p/E, where p is the net dipole moment of the sphere.)

II–4. Liquid helium-4 has a normal boiling point of $T_0 = 4.2K$ (at a pressure $p_0 =$ 1atm). However, at a lower pressure $p_1 = 1.3 \times 10^{-3}$atm, it boils at a lower temperature $T_1 = 1.2K$.

The Clapeyron-Clausius equation, describing the slope of the phase equilibrium line, is

$$\frac{dp}{dT} = \frac{\Delta s}{\Delta v} \qquad \text{(II–5)}$$

where Δs and Δv are the differences in molar entropy and molar volume between two phases, respectively. Assuming that gaseous helium is an ideal gas, use this equation to estimate the average latent heat of vaporization for temperatures between T_0 and T_1 in terms of p_0, p_1, T_0, T_1 and the gas constant R. Evaluate this value for the given pressures and temperatures.

I-1

a) Conservations of momentum and energy

$$\begin{cases} bu = bw + v \\ bu^2 = bw^2 + v^2 \end{cases}$$

lead to

$$\boxed{v = \frac{2b}{1+b}u}$$

This satisfies

$$\boxed{\frac{v}{u} \leq 2}$$

b)

$$v_{i+1} = \frac{2\lambda}{1+\lambda}v_i$$

leading to

$$\boxed{v_f = \left(\frac{2\lambda}{1+\lambda}\right)^n u}$$

for $\lambda > 1$, this can grow indefinitely if the number of particles is allowed to grow. For $\lambda = 4$, we need

$$(\frac{4}{3})^n > 20 \quad \Rightarrow \quad \boxed{n \geq 11}$$

I-2

a)

$$\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial t}|\psi|^2 = \frac{\partial}{\partial t}\psi^*\psi = \psi^*\frac{\partial \psi}{\partial t} + \frac{\partial \psi^*}{\partial t}\psi$$
$$= -i\psi^*\Big(-\frac{\nabla^2}{2m} + V\Big)\psi + i\psi\Big(-\frac{\nabla^2}{2m} + V\Big)\psi^*$$
$$= \frac{i}{2m}\boldsymbol{\nabla}\cdot(\psi^*\boldsymbol{\nabla}\psi - \psi\boldsymbol{\nabla}\psi^*)$$

This suggests the definition

$$\boxed{\mathbf{J} \equiv \frac{i}{2m}(\psi\boldsymbol{\nabla}\psi^* - \psi^*\boldsymbol{\nabla}\psi)}$$

b) In general, for

$$\psi = ae^{ikx} + be^{-ikx}$$

we have

$$\mathbf{J} = \frac{k}{m}(|a|^2 - |b|^2)\hat{\mathbf{x}}$$

In a stationary states, we have

$$\mathbf{J}\Big|_{-\infty}^{+\infty} = 0$$

equivalent to

$$|B|^2 - |A|^2 + |F|^2 - |G|^2 = 0$$

which is the same as saying that S is unitary. ■

I-3

a) Applying the divergence operator to

$$\nabla \times \mathbf{B} = \mu_0(\mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t})$$

gives

$$0 = \nabla \cdot \mathbf{J} + \varepsilon_0 \frac{\partial}{\partial t} \nabla \cdot \mathbf{E} = \nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t}$$

which is what we wanted to prove.

b) Assuming there are no currents at the edges of the universe, we have

$$\frac{dQ}{dt} = \frac{d}{dt} \int_{\text{universe}} \rho dV = \int_{\text{universe}} \frac{\partial \rho}{\partial t} dV = \int_{\text{universe}} \nabla \cdot \mathbf{J}\, dV = \oint_{\partial\,\text{universe}} \mathbf{J}.d\mathbf{a} = 0$$

c)

$$u \equiv \frac{1}{2}\varepsilon_0 E^2 + \frac{B^2}{2\mu_0}$$

$$\frac{\partial u}{\partial t} = \varepsilon_0 \mathbf{E}.\frac{\partial \mathbf{E}}{\partial t} + \frac{\mathbf{B}}{\mu_0}.\frac{\partial \mathbf{B}}{\partial t} = -\mathbf{J}.\mathbf{E} + \frac{1}{\mu_0}\big[\mathbf{E}.(\nabla \times \mathbf{B}) - \mathbf{B}.(\nabla \times \mathbf{E})\big]$$
$$= -\mathbf{E}.\mathbf{J} - \nabla \cdot \Big(\frac{\mathbf{E} \times \mathbf{B}}{\mu_0}\Big)$$

This simply means that the electromagnetic energy is not conserved on its own, but can be converted into heat/mechanical energy via the **E.J** term.

I-4

a) Let's assume that the input streams have maximum flow Q, then, for temperatures less than $(T_1 + T_2)/2$, the cold input is fully open and a fraction $w \in [0, 1]$ of the hot input is mixed with it. The fraction is given by

$$w = \frac{T - T_1}{T_2 - T}$$

Which means

$$Q_{\text{out}} = (1 + w)Q = \frac{T_2 - T_1}{T_2 - T} Q$$

in general,

$$\boxed{Q_{\text{out}} = \frac{T_2 - T_1}{\max(T_2 - T, T - T_1)}}$$

b) The maximum out flow is clearly less than $2Q$.

II-1

a) Let X be the displacement of the wedge to the left. Then, a careful evaluation of the geometry of the problem, leads to the following expressions for the kinetic and potential energy in terms of X

$$T = \frac{1}{2} m \Big(\frac{M + m}{m \cos\alpha}\Big)^2 \dot{X}^2$$

and

$$U = -(M + m) g X \tan\alpha.$$

Hence

$$\boxed{\dot{X} = gt\,\frac{m\sin\alpha\,\cos\alpha}{2M + m(1+\sin^2\alpha)}}$$

$$\boxed{X = \frac{1}{2}gt^2\,\frac{m\sin\alpha\,\cos\alpha}{2M + m(1+\sin^2\alpha)}}$$

II-2

a) This is a 1D problem with potential energy

$$V = \begin{cases} 0 & x \le 0 \\ \\ -e\phi & x > 0 \end{cases}$$

The suitable wavefunction is

$$\psi = \begin{cases} e^{ikx} + re^{-ikx} & x \le 0 \\ \\ te^{i\kappa x} & x > 0 \end{cases}$$

with

$$\kappa \equiv \sqrt{k^2 + 2me\phi}$$

$$r \equiv \frac{1-\kappa/k}{1+\kappa/k} \quad ; \quad t \equiv \frac{2}{1+\kappa/k}$$

b)

$$I_r = r^2 I_i \approx \boxed{60\,nA}$$

II-3

a) We already know that a constant polarization

$$\mathbf{P} = 3\varepsilon_0 E\hat{\mathbf{z}}$$

cancels the external uniform electric field. Outside the sphere, the potential, is the sum of a uniform field potential and a pure dipole term

$$\boxed{\phi = E\cos\theta\big(\frac{a^2}{r^2} - 1\big)}$$

b)

$$\boxed{\sigma = 3\varepsilon_0 E\cos\theta}$$

c)

$$\boxed{\alpha = 4\pi a^3\varepsilon_0}$$

II-4

First, since $v_g \gg v_l$, we approximate the relation by omitting v_l in the denominator. Next

$$v_g = \frac{RT}{p} \quad ; \quad \lambda = T(s_g - s_l)$$

The CC equation then becomes

$$\frac{dp}{dT} = \frac{\lambda p}{RT^2}$$

in other words

$$\lambda = R\frac{d(\log p)}{d(-1/T)} \approx R\frac{\Delta(\log p)}{\Delta(-1/T)} \approx \boxed{92.8\, J/mol}$$