PSU Physics PhD Qualifying Exam Solutions
Fall 2012

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Candidacy Exam
Department of Physics
October 6, 2012
Part I

Instructions:

- The following problems are intended to probe your understanding of basic physical principles. When answering each question, indicate the principles being applied and any approximations required to arrive at your solution. *If information you need is not given, you may define a variable or make a reasonable physical estimate, as appropriate.* Your solutions will be evaluated based on clarity of physical reasoning, clarity of presentation, and accuracy.
- Please use a new blue book for each question. Remember to write your name and the problem number of the cover of each book.
- We suggest you read all *four* of the problems before beginning to work them. You should reserve time to attempt every problem.

Fundamental constants, conversions, etc.:

Avogadro's number	N_A	$6.022 \times 10^{23}\,\mathrm{mol}^{-1}$
Boltzmann's constant	k_B	$1.381 \times 10^{-23}\,\mathrm{J\,K}^{-1}$
Electron charge magnitude	e	$1.602 \times 10^{-19}\,\mathrm{C}$
Gas constant	R	$8.314\,\mathrm{J\,mol}^{-1}\,\mathrm{K}^{-1}$
Planck's constant	h $\hbar = h/2\pi$	$6.626 \times 10^{-34}\,\mathrm{J\,s}$ $1.055 \times 10^{-34}\,\mathrm{J\,s}$
Speed of light in vacuum	c	$2.998 \times 10^{8}\,\mathrm{m\,s}^{-1}$
Permittivity constant	ϵ_0	$8.854 \times 10^{-12}\,\mathrm{F\,m}^{-1}$
Permeability constant	μ_0	$1.257 \times 10^{-6}\,\mathrm{N\,A}^{-2}$
Gravitational constant	G	$6.674 \times 10^{-11}\,\mathrm{m}^3\,\mathrm{kg}^{-1}\,\mathrm{s}^{-2}$
Standard atmospheric pressure	1 atmosphere	$1.01 \times 10^{5}\,\mathrm{N\,m}^{-2}$
Stefan-Boltzmann constant	σ	$5.67 \times 10^{-8}\,\mathrm{W\,m}^{-2}\,\mathrm{K}^{-4}$
Electron rest mass	m_e	$9.109 \times 10^{-31}\,\mathrm{kg} = 0.5110\,\mathrm{MeV}\,c^{-2}$
Proton rest mass	m_p	$1.673 \times 10^{-27}\,\mathrm{kg} = 938.3\,\mathrm{MeV}\,c^{-2}$
Origin of temperature scales		$0\,^\circ\mathrm{C} = 273\,\mathrm{K}$
1 large calorie (as in nutrition) 1 inch		$4.184\,\mathrm{kJ}$ $2.54\,\mathrm{cm}$

Definite integrals:

$$\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2} \ . \tag{I–1}$$

$$\int_0^\infty x^n e^{-x} dx = \Gamma(n+1) = n! \ . \tag{I–2}$$

$$\int_0^1 dx \sqrt{\frac{x}{1-x}} = \frac{\pi}{2} \ . \tag{I–3}$$

Laplacian in spherical polar coordinates (r, θ, ϕ):

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 f}{\partial \phi^2}. \tag{I–4}$$

Laplacian in cylindrical coordinates (r, θ, z):

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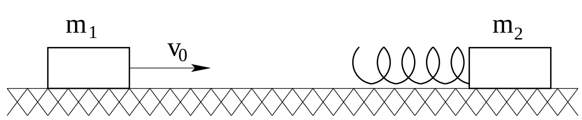
I–1. A frictionless sliding block with mass m_1 and translational velocity v_0 strikes a massless spring with spring constant k connected to a stationary frictionless block of mass $m_2 = m_1$ (see figure). A maximum compression of the spring occurs at the instant the velocities of both blocks are equal. If $m_1 = m_2 = 0.1\,\mathrm{kg}$ and $k = 6.5 \times 10^4\,\mathrm{N/m}$, what is the maximum compression of the spring when the initial translational velocity of the first block is $v_0 = 10\,\mathrm{m/s}$?



I–2. The correlation function of two operators $\mathcal{O}_1$ and $\mathcal{O}_2$ in some state $|s\rangle$ is defined to be the expectation value of the product of the two operators in this state,

$$\langle s|\mathcal{O}_1\mathcal{O}_2|s\rangle \ .$$

For a simple one-dimensional harmonic oscillator evaluate the correlation function of $\mathcal{O}_1 = x(t)$ and $\mathcal{O}_2 = x(0)$ in the ground state and the first excited state,

$$\langle 0|x(t)x(0)|0\rangle \quad \text{and} \quad \langle 1|x(t)x(0)|1\rangle \ .$$

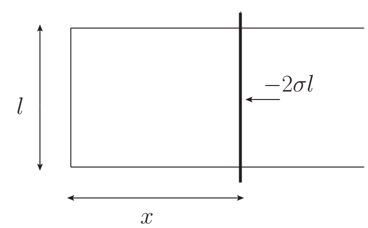
I–3. A soap film is supported by a wire frame which has one side of length l that can move without friction. The surface tension of the film exerts a force

$$F = 2\sigma l$$

on the loose side and its direction is such that it tends to decrease the area of the film. The surface tension σ depends on temperature as

$$\sigma = \sigma_0 - \alpha T \ ,$$

with σ_0 and α being constants independent of T or of the distance x between the moving side and the opposite one (see figure).



(a) Assuming that x is the only significant external parameter, write a relation expressing the change dE in the mean energy of the film in terms of the heat dQ absorbed by it and the work done by it in an infinitesimal quasistatic process in which x changes by dx.

(b) Compute the change in energy $\Delta E = E(x) - E(0)$ of the film when it is stretched from $x = 0$ to some length x at constant temperature $T = T_0$.

(c) Compute the work done on the film to stretch it at fixed temperature T_0 from $x = 0$ to some length x.

I–4. A charged particle of mass m and charge q is confined in a two dimensional harmonic oscillator potential, in the (x, y) plane:

$$V(x, y) = \frac{k}{2}(x^2 + y^2) \ .$$

(a) Determine the frequency of the oscillations.

(b) A magnetic field is applied in the z direction. Determine the new frequencies of oscillation for clockwise and counterclockwise rotation.

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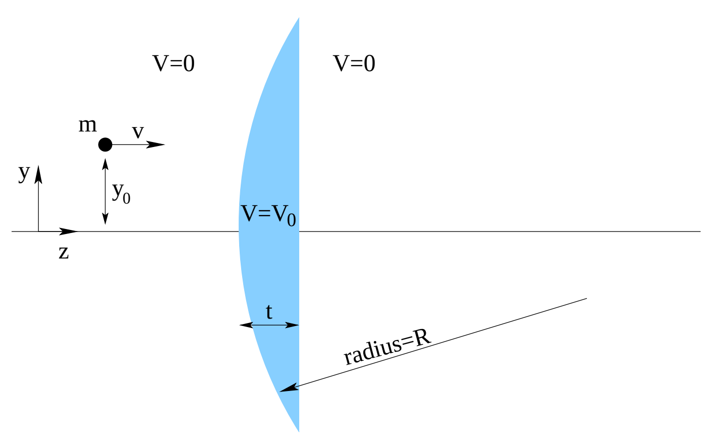
II–1. A non-relativistic particle of mass m constrained to move in the (y, z) plane moves freely except in a region where it encounters a constant potential, $V = V_0$. This region is bounded by a circle of radius R and a line at $z = 0$ as shown in the figure. The center of the circle is on the z axis. The thickness of the non-zero potential region is t. Consider the limit where $t \to 0$ $(t \ll R)$. The particle enters from the left, a distance y_0 from the axis and initial velocity v in the z direction. It passes through the non-zero potential region. Assume that the impulse from the potential change is directed perpendicular to the surfaces.



(a) Derive the equivalent of Snell's law for the change in direction when this particle passes through a surface with an abrupt change in potential.

(b) At what z position does the particle cross the z axis? Give the answer in terms of the mass m of the particle, its initial velocity v in the z direction, the potential V_0 in the bounded region, the radius R of the first surface and the initial position y_0 in the y direction.

II–2. P is a beam of atoms with spin-1/2 and zero orbital angular momentum, all with angular momenta $\hbar/2$ along the x axis. Q is a beam of similar but unpolarized atoms.

(a) What is the spin state function of P in terms of the eigenfunctions of S_z?

(b) If the two beams are passed though a Stern-Gerlach apparatus with its magnetic field along the z axis, is there any difference between the emerging beams in the two cases? Recall that the SternGerlach experiment involves sending a beam of particles through an inhomogeneous magnetic field and observing their deflection.

(c) How could the difference between the beams P and Q be detected experimentally?

II–3. The Clausius-Clapeyron equation expresses the effect of pressure on the melting temperature of ice, viz.

$$\frac{dp}{dT} = \frac{L}{(v_w - v_i)T} ,$$

where L is the latent heat of fusion (energy per unit mass) and v_i and v_w are the volumes of unit mass of ice and water at absolute temperature T.

Consider now a mass of 1 kg placed upon a block of ice at temperature $T = 273\,\mathrm{K}$. The weight bears down on an area of $1\,\mathrm{mm}^2$ causing the ice to melt; the latent heat is $L = 333\,\mathrm{kJ\,kg}^{-1}$. Assuming there is no heat transfer form this mass, determine by how much the temperature of the ice must be lowered for it to resist penetration by the mass.

Hint: The density of ice ρ_i in relation to the density of cold water ρ_w is related to the fact that ice floats 11/12 submerged.

II–4. Consider an infinite grounded conducting wall. An ion of charge q and mass m is placed at rest at distance a from it. The ion will be attracted towards the wall. Find the time T in which the ion reaches the wall.

I-1

Let's use the conservation laws:

$$\frac{1}{2}m_1v_0^2 = \frac{1}{2}(m_1+m_2)v_1^2 + \frac{1}{2}kx^2$$
$$m_1v_0 = (m_1+m_2)v_1$$

solved as

$$\boxed{x = \sqrt{\frac{m_1m_2v_0^2}{k(m_1+m_2)}} \approx 0.769\,mm}$$

I-2

$$\langle n|\, x(t)x(0)\,|n\rangle = \langle n|\, e^{iHt/\hbar}xe^{-iHt/\hbar}x\,|n\rangle = \langle n|\left(x\cos\omega t + \frac{p}{m\omega}\sin\omega t\right)x\,|n\rangle$$

For the x^2 term I use the virial theorem to find $\langle n|\, x^2\,|n\rangle = (n+1/2)\hbar/\omega m$. For the px term, it's best to write the operators down in terms of creation and annihilation operators

$$\langle n|\, x(t)x(0)\,|n\rangle = (n+\frac{1}{2})\frac{\hbar}{\omega m}\cos(\omega t) - \frac{i\hbar}{2m\omega}\sin\omega t\,\langle n|\,(a-a^\dagger)(a+a^\dagger)\,|n\rangle = \boxed{\frac{\hbar}{\omega m}\left[n\cos\omega t + \frac{1}{2}e^{-i\omega t}\right]}$$

I-3

a)

$$dE = đQ + 2\sigma l dx$$

b) The free energy is a function of state with

$$dF = -SdT + 2l(\sigma_0 - \alpha T)dx$$

leading to the Maxwell relation

$$\left(\frac{\partial S}{\partial x}\right)_T = 2l\alpha$$

integrated as

$$S = f(T) + 2l\alpha x$$

Now we may write the change in energy as

$$\Delta E = \int dE = \int (TdS + 2\sigma l dx) = 2l\sigma_0 \int dx = \boxed{2l\sigma_0 x}$$

c) The work is

$$W = \int 2\sigma l dx = \boxed{2l(\sigma_0 - \alpha T)x}$$

I-4

Define

$$\omega_0 \equiv \sqrt{k/m} \quad ; \quad \omega_B = \frac{qB}{2m}$$

a) $\boxed{\omega = \sqrt{k/m}}$

b) The constant magnetic field is given by $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$ with $\mathbf{A} = \frac{1}{2}Br\hat{\boldsymbol{\varphi}}$. The Hamiltonian is

$$\begin{aligned} H &= \frac{(i\boldsymbol{\nabla} + q\mathbf{A})^2}{2m} + \frac{1}{2}kr^2 \\ &= \frac{-\nabla^2}{2m} + \frac{iqB}{2m}\partial_\varphi + \frac{1}{2}(k + \frac{q^2B^2}{4m})r^2 \\ &= H_1 - \omega_B L_z \end{aligned}$$

Here H_1 is an isotropic 2D harmonic oscillator with $\omega_1 = \sqrt{\omega_0^2 + \omega_B^2}$. Being isotropic, it commutes with L_z, and therefore we have

$$U(t) \equiv \exp(-iHt) = \exp(-iH_1t - i\omega_B tL_z) = \exp(-i\omega_B tL_z)\exp(-iHt) = R_z(-\omega_B t)\exp(-iH_1t)$$

In other words, the time evolution of this system, differs from the harmonic oscillator H_1 by a constant rotation rate per time. Circular motions governed by H_1 have frequencies $\pm\omega_1$, and therefore, possible values of $d\langle\varphi\rangle/dt$ are

$$\boxed{\left.\frac{d\langle\varphi\rangle}{dt}\right|_\pm = -\omega_B \pm \sqrt{\omega_B^2 + \omega_0^2}}$$

or

$$\boxed{\omega_{\text{CW}} = \omega_B + \sqrt{\omega_B^2 + \omega_0^2} \quad ; \quad \omega_{\text{CCW}} = -\omega_B + \sqrt{\omega_B^2 + \omega_0^2}}$$

in accordance with the classical theory.

II-1

a) The transverse momentum is conserved

$$\boxed{v_i \sin\theta_i = v_r \sin\theta_r}$$

b) The effective index of refraction is found using the conservation of energy

$$n \equiv \frac{v_{\text{out}}}{v_{\text{in}}} = \frac{1}{\sqrt{1 - 2V_0/mv^2}}$$

The first incident angle is

$$\theta_i = \arcsin\frac{y_0}{R}$$

Then, the refraction tilts the direction of the particle by an amount

$$\psi_1 = \theta_i - \theta_r = \arcsin\frac{y_0}{R} - \arcsin\frac{y_0}{nR}$$

The final tilt angle is

$$\psi = \arcsin\left[n\sin\left(\arcsin\frac{y_0}{R} - \arcsin\frac{y_0}{nR}\right)\right]$$

The particle crosses the z axis at

$$z = \frac{y_0}{\tan\psi} = \boxed{y_0\cot\left\{\arcsin\left[n\sin\left(\arcsin\frac{y_0}{R} - \arcsin\frac{y_0}{nR}\right)\right]\right\}}$$

II-2

a)

$$\boxed{|P\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle + |\downarrow\rangle\right)}$$

b) No; both beams split in half.

c) By measuring the spin angular momentum along any direction that is not perpendicular to the x axis. It is most vivisble by measuring the angular momentum along the x axis where all of the atoms from P end up in one slot while the Q beam splits in half again.

II-3

$$\delta T = \frac{\delta p \times T \times (v_w - v_i)}{L} = \frac{MgTv_w}{AL} \times (1 - \frac{12}{11}) \approx \boxed{-0.73\,K}$$

II-4

[1]

The attractive force at a distance x from the wall is

$$-m\ddot{x} = \frac{q^2}{16\pi\varepsilon_0 x^2}$$

or

$$\dot{x}\frac{d\dot{x}}{dx} = -\frac{q^2}{16\pi\varepsilon_0 m}\frac{1}{x^2}$$

integrated as

$$\frac{dx}{dt} = \frac{-q}{\sqrt{8\pi\varepsilon_0 m}}\sqrt{\frac{1}{x} - \frac{1}{a}}$$

Now the time can be written as an integral

$$T = \frac{\sqrt{8\pi\varepsilon_0 m}}{q}a^{3/2}\int_0^1 \frac{dx}{\sqrt{1/x - 1}} = \boxed{\sqrt{\frac{2\pi^3\varepsilon_0 m a^3}{q^2}}}$$

[1] There is a way to solve this without using calculus as well. Cf. Kepler's law!