PSU Physics PhD Qualifying Exam Solutions
Spring 2012

Koorosh Sadri

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Candidacy Exam
Department of Physics
February 4, 2012
Part I

Instructions:

- The following problems are intended to probe your understanding of basic physical principles. When answering each question, indicate the principles being applied and any approximations required to arrive at your solution. *If information you need is not given, you may define a variable or make a reasonable physical estimate, as appropriate.* Your solutions will be evaluated based on clarity of physical reasoning, clarity of presentation, and accuracy.
- Please use a new blue book for each question. Remember to write your name and the problem number of the cover of each book.
- We suggest you read all *four* of the problems before beginning to work them. You should reserve time to attempt every problem.

Fundamental constants, conversions, etc.:

Avogadro's number	N_A	$6.022 \times 10^{23}\,\mathrm{mol^{-1}}$
Boltzmann's constant	k_B	$1.381 \times 10^{-23}\,\mathrm{J\,K^{-1}}$
Electron charge magnitude	e	$1.602 \times 10^{-19}\,\mathrm{C}$
Gas constant	R	$8.314\,\mathrm{J\,mol^{-1}\,K^{-1}}$
Planck's constant	h	$6.626 \times 10^{-34}\,\mathrm{J\,s}$
	$\hbar = h/2\pi$	$1.055 \times 10^{-34}\,\mathrm{J\,s}$
Speed of light in vacuum	c	$2.998 \times 10^{8}\,\mathrm{m\,s^{-1}}$
Permittivity constant	ϵ_0	$8.854 \times 10^{-12}\,\mathrm{F\,m^{-1}}$
Permeability constant	μ_0	$1.257 \times 10^{-6}\,\mathrm{N\,A^{-2}}$
Gravitational constant	G	$6.674 \times 10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}}$
Standard atmospheric pressure	1 atmosphere	$1.01 \times 10^{5}\,\mathrm{N\,m^{-2}}$
Stefan-Boltzmann constant	σ	$5.67 \times 10^{-8}\,\mathrm{W\,m^{-2}\,K^{-4}}$
Electron rest mass	m_e	$9.109 \times 10^{-31}\,\mathrm{kg} = 0.5110\,\mathrm{MeV}\,c^{-2}$
Proton rest mass	m_p	$1.673 \times 10^{-27}\,\mathrm{kg} = 938.3\,\mathrm{MeV}\,c^{-2}$
Origin of temperature scales		$0\,^\circ\mathrm{C} = 273\,\mathrm{K}$
1 large calorie (as in nutrition)		$4.184\,\mathrm{kJ}$
1 inch		$2.54\,\mathrm{cm}$

Definite integrals:

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}. \tag{I–1}$$

$$\int_{0}^{\infty} x^n e^{-x} dx = \Gamma(n+1) = n!. \tag{I–2}$$

Laplacian in spherical polar coordinates (r, θ, ϕ):

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 f}{\partial \phi^2}. \tag{I–3}$$

Laplacian in cylindrical coordinates (r, θ, z):

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I–1. A body of mass m moves in one dimension under the influence of a conservative force with potential energy $U(x)$.

(a) Show that when this body is displaced a small amount from its position of stable equilibrium $x = x_0$, it experiences a restoring force with force constant $\left[\frac{d^2U}{dx^2}\right]_{x=x_0}$.

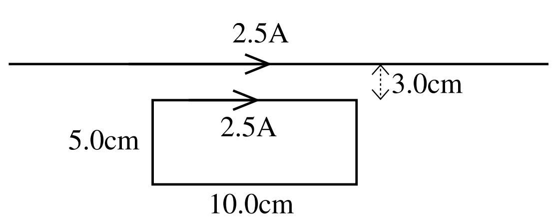
(b) Let the potential energy have the form

$$U(x) = \frac{-cx}{x^2 + a^2}, \tag{I–5}$$

where a and c are positive constants. Find the position of stable equilibrium and calculated the angular frequency of the oscillations.

(c) Sketch the form of this potential and the force resulting from it as a function of x.

I–2. A rectangular loop of wire is placed next to a long straight piece of similar wire, each carrying a steady current 2.5 A in the directions shown:



Find (a) the magnitude, and (b) the direction of the net force acting on the loop.

I–3. A spin-half particle has magnetic moment μ and is placed in a constant magnetic field B that points in the z direction. What is an appropriate matrix representation of the three spin operators S_x S_y, and S_z, and why? What is the quantum-mechanical Hamiltonian for the spin state of the system?

Initially, at time $t = 0$, the particle's spin is in $+x$ direction. Find the time-dependence of the spin state of the particle as a function of t, in a basis of eigenstates of S_z.

If the y-component of spin is measured at a time t_1, what are the possible outcomes and their probabilities?

I–4. An ideal gas is contained in a large jar of volume V_0. Fitted to the top of the jar is a tube of cross sectional area A in which a ball of mass m fits snugly and can slide without friction. The tube points upwards, and the jar is otherwise sealed.

Due to the weight of the ball the equilibrium pressure in the jar is slightly higher than the atmospheric pressure p_0. If the ball is displaced slightly it will execute simple harmonic oscillator motion. Assuming that, from the perspective of the gas, this motion is adiabatic and defining $\gamma = C_p/C_v$, find a relation between the oscillation frequency and γ, A, p_0, m, V_0. In your solution, you should take V_0 to be the volume of gas when the ball is at the equilibrium point.

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II–1. A small meteor approaches Earth with impact parameter b and velocity at infinity v_∞. Find, as a function of the radius of the Earth a, of v_∞, and of the escape velocity v_0, the smallest impact parameter for which the meteor will not strike the Earth. The escape velocity is defined as the velocity of a particle such that it has zero total energy in the gravitational field of the Earth.

II–2. Two identical parallel metallic plates each have an area A. Their vacuum gap is d.

(a) What is the capacitance?

(b) If a voltage V is applied between the gaps, what is the stored energy?

(c) How does the stored energy change if the vacuum gap is replaced by a material with dielectric constant $\epsilon > \epsilon_0$, while the capacitor is kept at constant V?

(d) If the dielectric material is only halfway into the gap, is the force on it pulling into the gap, or out of it?

II–3. A particle of mass m moves in one dimension. It is found that the exact eigenfunction for its quantum mechanical ground state is

$$\psi(x) = \frac{A}{\cosh(\lambda x)}, \tag{II–5}$$

where A and λ are constants. Assuming that the potential function $V(x)$ vanishes at infinity ($|x| \to \infty$), what are: (a) the ground state energy eigenvalue, and (b) $V(x)$?

II–4. (a) Consider an ideal gas of nitrogen molecules (N_2) at temperature $T = 300\,\mathrm{K}$. What is the root-mean-square speed of the molecules, given that their mass is $2 \times 14 \times 1.67 \times 10^{-27}\,\mathrm{kg}$?

(b) Now suppose spherical droplets of water, of diameter 1.2 microns are introduced into this gas at this same temperature, such that equilibrium is maintained. What is the root-mean-square speed of the droplets, given that the density of water is $1.0 \times 10^3\,\mathrm{kg/m^3}$?

I-1

a)

$$F = -U'(x) = -U'(x_0) - (x - x_0)U''(x_0) - \cdots \approx -k(x - x_0)$$

with

$$\boxed{k = U''(x_0)}$$

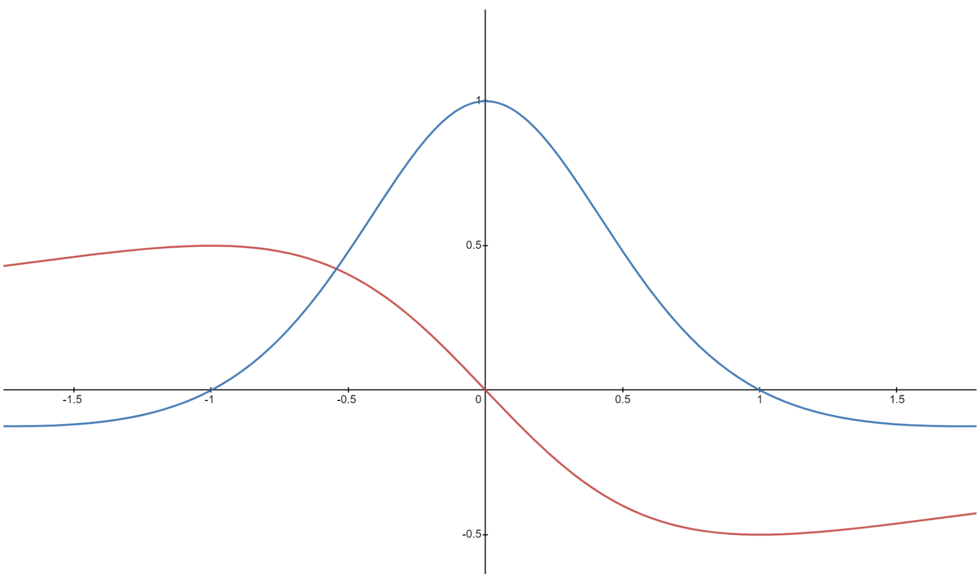
b) The equilibrium in $x < 0$ is unstable. Instead, there is a stable equilibrium at

$$U'(x_0 > 0) = 0 \;\Leftrightarrow\; x_0 = a$$

The frequency is

$$\omega = \sqrt{\frac{k}{m}} = \boxed{\sqrt{\frac{c}{2ma^3}}}$$

c) The curves are in units where $c = a = 1$. The red curve is the potential and the blue curve is the force.



I-2

a, b) Let $\hat{\mathbf{z}}$ point in the right direction. Then

$$\mathbf{F} = I_{\text{loop}} \oint_{\text{loop}} d\mathbf{l} \times \mathbf{B}_{\text{wire}} = \frac{\mu_0}{2\pi} I_{\text{loop}} I_{\text{wire}} \oint_{\text{loop}} d\mathbf{l} \times \frac{\hat{\boldsymbol{\varphi}}}{s} = \frac{\mu_0}{2\pi} I_{\text{loop}} I_{\text{wire}} \left(-\frac{10}{3}\hat{\mathbf{s}} + \frac{10}{8}\hat{\mathbf{s}} \right) \approx \boxed{2.60 \times 10^{-6}\, N}$$

I-3

A representation for the proper matrices are

$$\mathbf{S} = \frac{\hbar}{2}\left[\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, \begin{pmatrix} & -i \\ i & \end{pmatrix}, \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \right]$$

because they satisfy the algebra

$$[S_i, S_j] = i\hbar\varepsilon_{ijk}S_k$$

and $\sum_i S_i^2 = \frac{3}{4}\hbar^2$

The Hamiltonian is

$$H = -\boldsymbol{\mu}.\mathbf{B} = -\gamma B S_z = -\frac{\gamma B\hbar}{2}\begin{pmatrix}1 & \\ & -1\end{pmatrix}$$

where $\gamma = \mu/S$ is the gyromagnetic ratio of the particle.

From this, we may find the unitary evolution operator as

$$U(t) \equiv \exp(-iHt/\hbar) = \begin{pmatrix}\exp(i\gamma Bt/2) & \\ & \exp(-i\gamma Bt/2)\end{pmatrix}$$

therefore

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}U(t)\begin{pmatrix}1\\1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}\exp(+i\gamma Bt/2)\\ \exp(-i\gamma Bt/2)\end{pmatrix}$$

The outcome will be in $\{\pm 1\}$, with

$$P_+ = |\,\langle +, y|\psi(t)\rangle\,|^2 = |\frac{1}{2}(1 \quad i)\begin{pmatrix}\exp(+i\gamma Bt/2)\\ \exp(-i\gamma Bt/2)\end{pmatrix}|^2 = \frac{1}{2}\left[1 + \sin(\gamma Bt)\right]$$

and

$$P_- = 1 - P_+ = \frac{1}{2}\left[1 - \sin(\gamma Bt)\right]$$

I-4

The equation of motion is

$$m\ddot{z} = A\delta p = -A\gamma p_{\text{eq.}}\frac{\delta V}{V} = -\frac{\gamma A^2}{V}(p_0 + \frac{mg}{A})z$$

Leading to

$$\boxed{\omega^2 = \frac{\gamma A^2}{mV}(p_0 + \frac{mg}{A})}$$

II-1

Let's start by asking what the nearest distance to the Earth will be, assuming no collision happens. At the nearest distance r_p, we may impose the conservation laws of energy and angular momentum

$$r_p v_p = v_\infty b \quad ; \quad \frac{1}{2}v_p^2 - \frac{GM}{r_p} = \frac{1}{2}v_\infty^2$$

solving for r_p gives

$$r_p = -\frac{GM}{v_\infty^2} + \sqrt{\left(\frac{GM}{v_\infty^2}\right)^2 + b^2}$$

On the other hand, we have $GM = \frac{1}{2}av_0^2$. Therefore, the no-collision condition becomes

$$a < -\frac{av_0^2}{2v_\infty^2} + \sqrt{b^2 + \left(\frac{av_0^2}{2v_\infty^2}\right)^2}$$

equivalent to

$$\boxed{b > a\sqrt{1 + (\frac{v_0}{v_\infty})^2}}$$

II-2

a)

$$\boxed{C = \frac{\varepsilon_0 A}{d}}$$

b)

$$\boxed{U = \frac{1}{2} C V^2 = \frac{\varepsilon_0 A}{2d} V^2}$$

c) It increases by a factor $\varepsilon/\varepsilon_0$

d) If one slowly lets the dielectric go in, the capacity increases an amount ΔC. The batteries therefore pump a charge $\Delta Q = V\Delta C$ to the capacitor, doing a work $V^2\Delta C$. But the energy increases only by $\frac{1}{2}V^2\Delta C$, therefore the person holding the dielectric has done negative work, meaning the force was pulling the dielectric inside throughout.

II-3

a, b) Firs of all, the wave function has no roots, meaning that it is the ground state. It decays exponentailly as $\exp(-\lambda x)$, suggesting a binding energy

$$\boxed{E_g = -\frac{\hbar^2\lambda^2}{2m}}$$

The potential is

$$V(x) = \frac{\hbar^2}{2m}\left(\frac{\psi''}{\psi} - \lambda^2\right) = \boxed{-\frac{\hbar^2\lambda^2}{m}\sec^2(\lambda x)}$$

II-4

a)

$$v_{\text{r.m.s.}} = \sqrt{\frac{3k_B T}{m}} \approx 515\, m/s$$

b)

$$v_{\text{r.m.s.}} = \sqrt{\frac{3k_B T}{m}} \approx 3.71\, mm/s$$