PSU Physics PhD Qualifying Exam Solutions
Fall 2013

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Candidacy Exam
Department of Physics
October 5, 2013

Part I

Instructions:

- The following problems are intended to probe your understanding of basic physical principles. When answering each question, indicate the principles being applied and any approximations required to arrive at your solution. *If information you need is not given, you may define a variable or make a reasonable physical estimate, as appropriate.* Your solutions will be evaluated based on clarity of physical reasoning, clarity of presentation, and accuracy.
- Please use a new blue book for each question. Remember to write your name and the problem number of the cover of each book.
- We suggest you read all *four* of the problems before beginning to work them. You should reserve time to attempt every problem.

Fundamental constants, conversions, etc.:

Avogadro's number	N_A	$6.022 \times 10^{23}\,\mathrm{mol}^{-1}$
Boltzmann's constant	k_B	$1.381 \times 10^{-23}\,\mathrm{J\,K}^{-1}$
Electron charge magnitude	e	$1.602 \times 10^{-19}\,\mathrm{C}$
Gas constant	R	$8.314\,\mathrm{J\,mol}^{-1}\,\mathrm{K}^{-1}$
Planck's constant	h $\hbar = h/2\pi$	$6.626 \times 10^{-34}\,\mathrm{J\,s}$ $1.055 \times 10^{-34}\,\mathrm{J\,s}$
Speed of light in vacuum	c	$2.998 \times 10^{8}\,\mathrm{m\,s}^{-1}$
Permittivity constant	ϵ_0	$8.854 \times 10^{-12}\,\mathrm{F\,m}^{-1}$
Permeability constant	μ_0	$1.257 \times 10^{-6}\,\mathrm{N\,A}^{-2}$
Gravitational constant	G	$6.674 \times 10^{-11}\,\mathrm{m}^3\,\mathrm{kg}^{-1}\,\mathrm{s}^{-2}$
Standard atmospheric pressure	1 atmosphere	$1.01 \times 10^{5}\,\mathrm{N\,m}^{-2}$
Stefan-Boltzmann constant	σ	$5.67 \times 10^{-8}\,\mathrm{W\,m}^{-2}\,\mathrm{K}^{-4}$
Electron rest mass	m_e	$9.109 \times 10^{-31}\,\mathrm{kg} = 0.5110\,\mathrm{MeV}\,c^{-2}$
Proton rest mass	m_p	$1.673 \times 10^{-27}\,\mathrm{kg} = 938.3\,\mathrm{MeV}\,c^{-2}$
Origin of temperature scales		$0\,^\circ\mathrm{C} = 273\,\mathrm{K}$
1 large calorie (as in nutrition) 1 inch		$4.184\,\mathrm{kJ}$ $2.54\,\mathrm{cm}$

Definite integrals:

$$\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}. \tag{I–1}$$

$$\int_0^\infty x^n e^{-x} dx = \Gamma(n+1) = n!. \tag{I–2}$$

Transformation of Lorentz 4-vector (e.g., $(ct, \mathbf{x})$) under boost by velocity v in z direction:

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{1-v^2/c^2}} & 0 & 0 & \frac{v/c}{\sqrt{1-v^2/c^2}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{v/c}{\sqrt{1-v^2/c^2}} & 0 & 0 & \frac{1}{\sqrt{1-v^2/c^2}} \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \tag{I–3}$$

Laplacian in spherical polar coordinates (r, θ, ϕ):

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 f}{\partial \phi^2}. \tag{I–4}$$

Laplacian in cylindrical coordinates (r, θ, z):

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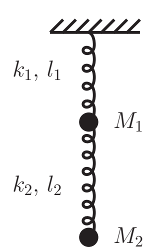
I–1. Two springs are hung in series from a support point. The top spring has spring constant k_1 and equilibrium length l_1, and the bottom spring has spring constant k_2 and equilibrium length l_2. At the junction of the springs is a mass M_1 and at the bottom is a mass M_2. Use classical mechanics, and ignore the internal dynamics and mass of the springs.

Find the equilibrium positions of the masses.

Find the possible angular frequencies of vertical sinusoidal oscillations about the equilibrium point.



I–2. Two conducting hemispheres are joined into a single sphere with a thin layer of electrical insulation to form a sphere of radius R with the center at the origin. The resulting sphere has the following surface potential that depends on the polar angle θ relative to the $\hat{z}$ axis:

$$V(R,\theta) = \begin{cases} V_0 & \text{for } \cos\theta > 0, \\ -V_0 & \text{for } \cos\theta < 0. \end{cases} \tag{I–6}$$

Assume that the electric potential at polar coordinates (r, θ, ϕ) outside the sphere is

$$V(r,\theta) = \sum_{L=0}^{\infty} \frac{a_L}{r^{L+1}} P_L(\cos\theta), \tag{I–7}$$

where the P_L are Legendre polynomials. We provide the following:

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{2}(3x^2 - 1). \tag{I–8}$$

(a) Determine the first three a_L (i.e., for $L = 0, 1, 2$).

(b) The electric field in the $\cos\theta = 0$ plane is given by $\mathbf{E} = E_r\hat{r} + E_z\hat{z}$, where $\hat{r}$ and $\hat{z}$ are unit vectors in the radial and z directions. Find expressions for E_r and E_z suitable for $r \gg R$. Keep only the leading (non-vanishing) power of $1/r$ for each component.

I–3. A particle of mass m in a one dimensional quantum-mechanical system has the normalized wave function

$$\Psi(x,t) = Ae^{-a(mx^2/\hbar + it)} \tag{I–9}$$

where A and a are positive real constants.

(a) Find A in terms of the other parameters a, m, and $\hbar$.

(b) Find the potential energy function $V(x)$ for which $\Psi(x,t)$ satisfies the Schrödinger equation.

(c) Compute the standard deviation of x in this state.

I–4. Assume that the lower 10 km of the atmosphere is in a convective steady state with constant entropy, so that for parcels of air in convection PV^γ is a constant, where $\gamma = C_P/C_V$, which is 1.4 for air. The molar mass of air is $\mu = 29$ g/mole.

(a) Find a formula for the change of temperature with altitude: dT/dz.

(b) Estimate its value in K/km.

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II–1. A neutral pion has an average lifetime of 8.4×10^{-17} s, decaying to two massless photons. The pion has mass $m_\pi = 0.135\,\mathrm{GeV}/c^2 = 2.41 \times 10^{-28}$ kg. The relation between energy and momentum is $E^2 = p^2c^2 + m^2c^4$. The photon is massless and thus has the energy momentum relationship of $E^2 = p^2c^2$.

(a) In the rest frame of a pion, one photon is emitted in the $\hat{z}$ direction. What is the energy of this photon as observed in the rest frame in GeV?

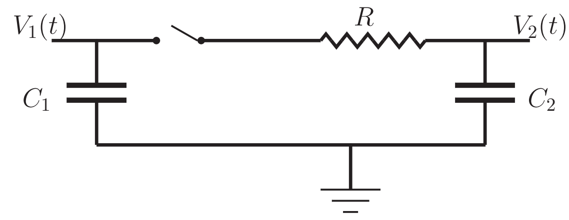
(b) If the pion was part of a beam of pions moving in the lab frame along the direction before decaying, with a total energy of $1\,\mathrm{GeV} = 10^9\,\mathrm{eV}$, what would the average lifetime of these pions be as seen in the lab frame?

(c) Again for a beam of pions with total energy of 1 GeV, calculate the energy of the forward decay photon discussed in part (a) when observing in the lab frame?

(d) Based on energy conservation, what must the energy of the other photon be in the lab frame?

II–2. In the circuit below, the capacitors labeled C_1 and C_2 both have capacitance C. They are connected through a switch and a resistor of resistance R. Capacitor C_1 is initially charged with a potential $V_1(0)$ at time $t = 0$, while capacitor C_2 is uncharged at time t=0.

At time $t = 0$, the switch is closed and the charge on C_1 begins to move to C_2. Write a differential equation for the time dependence of $V_1(t)$ and $V_2(t)$. Solve for the time dependence of the voltage across C_2 for times after $t = 0$ when the switch is closed.



II–3. Consider a system which has only two linearly-independent states; choose them to be

$$|1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{II–6}$$

Assume furthermore that, in this basis, the Hamiltonian of the system is

$$H = \begin{pmatrix} h & g \\ g & h \end{pmatrix}, \tag{II–7}$$

where g and h are nonzero real constants.

(a) Find normalized eigenvectors of H.

(b) Assume that the system starts at $t = 0$ in state $|1\rangle$. Find the state at time $t > 0$.

II–4. A zipper has N links. Each link has two possible states: (i) a closed state with energy zero, and (ii) an open state of energy ε. The zipper only unzips from one side, and a link can only be open if all links to its left are also open.

(a) Find the partition function.

(b) Find the average number of open links as a function of temperature.

I-1

In equilibrium, the tension in the first spring supports the weight of both masses, therefore

$$l_1^{(eq)} = l_1 + \frac{g}{k_1}(M_1 + M_2)$$

while the second spring is only supporting the lower mass

$$l_2^{(eq)} = l_2 + \frac{M_2 g}{k_1}$$

For small oscillations, if x_i denotes the displacement of the ith mass from its equilibrium point, we have

$$\frac{d^2}{dt^2}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -\frac{k_1+k_2}{M_1} & \frac{k_2}{M_1} \\ \frac{k_2}{M_2} & \frac{-k_2}{M_2} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

The characteristic equation reads

$$\omega^4 - \left(\frac{k_1+k_2}{M_1} + \frac{k_2}{M_2}\right)\omega^2 + \frac{k_1 k_2}{M_1 M_2} = 0$$

solved as

$$\boxed{\omega_\pm^2 = \frac{1}{2}\left(\frac{k_1+k_2}{M_1} + \frac{k_2}{M_2}\right) \pm \sqrt{\frac{1}{4}\left(\frac{k_1+k_2}{M_1} + \frac{k_2}{M_2}\right)^2 - \frac{k_1 k_2}{M_1 M_2}}}$$

I-2

a) At the surface we have

$$V_0 \,\mathrm{sgn}(x) = \sum_l \frac{a_l}{R^{l+1}} P_l(x)$$

The parity itself shows that the even coefficients vanish ($\boxed{a_0 = a_2 = 0}$). As for a_1:

$$v_0 \int_0^1 x dx = \frac{a_1}{R^2}\int_0^1 x^2 dx \quad \Rightarrow \quad \boxed{a_1 = \frac{3R^2 V_0}{2}}$$

b) The leading order potential is the dipole potential

$$V = \frac{3}{2}R^2 V_0 \frac{\cos\theta}{r^2} \quad \Rightarrow \quad \mathbf{E} = \frac{3R^2 V_0}{2r^3}\left(2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\boldsymbol{\theta}}\right) = \boxed{\frac{3R^2 V_0}{2r^3}\left(3\cos\theta\hat{\mathbf{r}} - \hat{\mathbf{z}}\right)}$$

I-3

a)

$$\boxed{A = \left(\frac{2am}{\pi\hbar}\right)^{1/4}}$$

b) Substituting the wave function in the Schroedinger equation, one reads

$$\boxed{V(x) = 2a\left(max^2 - \hbar\right)}$$

c)

$$\boxed{\sigma^2 = \frac{\hbar}{4ma}}$$

I-4

a, b) Assuming ideal gas:

$$\frac{dT}{dz} = \frac{\mu}{R}\left(\frac{1}{\rho} - \frac{p}{\rho^2}\frac{\partial\rho}{\partial p}\Big|_{ad.}\right)\frac{dp}{dz} = -\frac{(\gamma-1)g\mu}{\gamma R} \approx 9.8\,K/km$$

II-1

a)

$$E = \frac{1}{2}m_\pi c^2 = 0.0675\,GeV$$

b)

$$\tau = \gamma\tau_0 = (E/m_\pi c^2)\tau_0 = 6.22\times 10^{-16}\,s$$

c, d) The two conservation laws read

$$E_+ + E_- = \gamma mc^2$$
$$E_+ - E_- = \sqrt{\gamma^2-1}mc^2$$

Therefore

$$E_+ = \frac{1}{2}\left(\gamma + \sqrt{\gamma^2-1}\right)mc^2 \approx 0.995\,GeV$$

and

$$E_- \approx 4.58\,MeV$$

II-2

The KCL gives $C(\dot{V}_1 + \dot{V}_2) = 0$; in addition to the initial conditions, this gives $V_1(t) = V_1(0) - V_2(t)$. The KVL reads

$$C\frac{dV_2}{dt} = \frac{1}{R}\left[V_1(t) - V_2(t)\right] = \frac{1}{R}\left[V_1(0) - 2V_2(t)\right]$$

solved as

$$\boxed{V_2(t>0) = \frac{V_1(0)}{2}\left(1 - e^{-2t/RC}\right)}$$

II-3

a)

$$\boxed{H\begin{pmatrix}+1/\sqrt{2}\\ \pm 1/\sqrt{2}\end{pmatrix} = (h \pm g)\begin{pmatrix}+1/\sqrt{2}\\ \pm 1/\sqrt{2}\end{pmatrix}}$$

b)

$$|\psi(t)\rangle = \frac{e^{-iht/\hbar}}{\sqrt{2}}\left(e^{-igt/\hbar}\,|+\rangle + e^{+igt/\hbar}\,|-\rangle\right)$$

$$= \frac{e^{-iht/\hbar}}{2}\left[e^{-igt/\hbar}\begin{pmatrix}1\\1\end{pmatrix} + e^{+igt/\hbar}\begin{pmatrix}1\\-1\end{pmatrix}\right] = e^{-iht/\hbar}\begin{pmatrix}\cos\, gt/\hbar\\ -i\sin\, gt/\hbar\end{pmatrix}$$

II-4

a)

$$Z = \sum_{n=0}^{N} e^{-\beta\varepsilon n} = \frac{1-e^{-(N+1)\beta\varepsilon}}{1-e^{-\beta\varepsilon}}$$

b)

$$\langle n\rangle = \frac{1-e^{-\beta\varepsilon}}{1-e^{-(N+1)\beta\varepsilon}}\sum_{n=0}^{N} n e^{-n\beta\varepsilon}$$

$$= \frac{e^{-\beta\varepsilon}}{(1-e^{-\beta\varepsilon})(1-e^{-(N+1)\beta\varepsilon})}\left[1-(N+1)e^{-N\beta\varepsilon} + Ne^{-(N+1)\beta\varepsilon}\right]$$