PSU Physics PhD Qualifying Exam Solutions
Spring 2013

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July 9, 2023

Qualifying Exam for Ph.D. Candidacy
Department of Physics
February 2nd, 2013

Part I

Instructions:

- The following problems are intended to probe your understanding of basic physical principles. When answering each question, indicate the principles being applied and any approximations required to arrive at your solution. *If information you need is not given, you may define a variable or make a reasonable physical estimate, as appropriate.* Your solutions will be evaluated based on clarity of physical reasoning, clarity of presentation, and accuracy.
- Please use a new blue book for each question. Remember to write your name and the problem number of the cover of each book.
- We suggest you read all *four* of the problems before beginning to work them. You should reserve time to attempt every problem.

Fundamental constants, conversions, etc.:

Avogadro's number	N_A	$6.022 \times 10^{23}\,\mathrm{mol^{-1}}$
Boltzmann's constant	k_B	$1.381 \times 10^{-23}\,\mathrm{J\,K^{-1}}$
Electron charge magnitude	e	$1.602 \times 10^{-19}\,\mathrm{C}$
Gas constant	R	$8.314\,\mathrm{J\,mol^{-1}\,K^{-1}}$
Planck's constant	h	$6.626 \times 10^{-34}\,\mathrm{J\,s}$
	$\hbar = h/2\pi$	$1.055 \times 10^{-34}\,\mathrm{J\,s}$
Speed of light in vacuum	c	$2.998 \times 10^{8}\,\mathrm{m\,s^{-1}}$
Permittivity constant	ϵ_0	$8.854 \times 10^{-12}\,\mathrm{F\,m^{-1}}$
Permeability constant	μ_0	$1.257 \times 10^{-6}\,\mathrm{N\,A^{-2}}$
Gravitational constant	G	$6.674 \times 10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}}$
Standard atmospheric pressure	1 atmosphere	$1.01 \times 10^{5}\,\mathrm{N\,m^{-2}}$
Stefan-Boltzmann constant	σ	$5.67 \times 10^{-8}\,\mathrm{W\,m^{-2}\,K^{-4}}$
Electron rest mass	m_e	$9.109 \times 10^{-31}\,\mathrm{kg} = 0.5110\,\mathrm{MeV}\,c^{-2}$
Proton rest mass	m_p	$1.673 \times 10^{-27}\,\mathrm{kg} = 938.3\,\mathrm{MeV}\,c^{-2}$
Origin of temperature scales		$0\,^\circ\mathrm{C} = 273\,\mathrm{K}$
1 large calorie (as in nutrition)		$4.184\,\mathrm{kJ}$
1 inch		$2.54\,\mathrm{cm}$

Definite integrals:

$$\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}. \tag{I–1}$$

$$\int_0^\infty x^n e^{-x} dx = \Gamma(n+1) = n!. \tag{I–2}$$

Laplacian in spherical polar coordinates (r, θ, ϕ):

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 f}{\partial \phi^2}. \tag{I–3}$$

Laplacian in cylindrical coordinates (r, θ, z):

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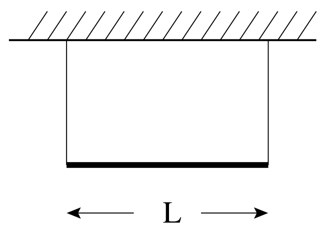
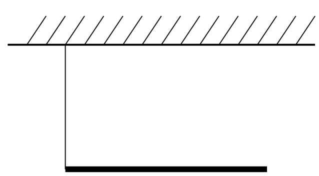
I–1. Initially a rigid bar is suspended by vertical inextensible light strings at its ends, so that the bar is horizontal:



The bar is of length L, its mass m is uniformly distributed along its length, and its width is negligible.

(a) Find the moment of inertia of the bar about its center and about its end.

(b) The string on the right is cut:



Find the tension in the remaining string very shortly after the other string is cut.

I–2. Consider a system of two spin-1/2 particles interacting through the Hamitonian

$$H = A(S_x^2 - S_y^2) + BS_z^2 \,, \tag{I–5}$$

where A and B are constants and S_x, S_y and S_z are the three components of the total spin of the system. Find the energy spectrum and the corresponding eigenvectors.

I–3. A plane-parallel 15nF capacitor is connected across 70V battery. How much work must be done to double the plate separation

(a) with the battery connected?

(b) with the battery disconnected?

I–4. A box of volume V_0 has a small hole of area A_0. The box initially has one mole of an ideal gas at $t = 0$, which is at an initial temperature $T(t = 0)$. Find the rate of energy flow through the hole as a function of temperature and other parameters.

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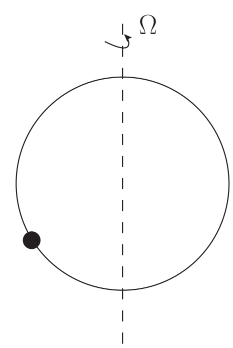
II–1. Consider a particle that can move freely along a circular hoop that rotates about a vertical axis with angular velocity Ω. The radius of the hoop is R.

(a) Find the Lagrangian.

(b) Find the equations of motion and determine the equilibrium position(s) of the particle.

(c) Determine which of the equilibrium position(s) is stable, and under what conditions.



II–2. A particle of mass m is confined to a one-dimensional region $0 \leq x \leq a$. At time $t = 0$, its normalized wave function is

$$\psi(x, t=0) = \sqrt{\frac{32}{17a}}\left[1 + \frac{1}{2}\cos\left(\frac{2\pi x}{a}\right)\right]\sin\left(\frac{2\pi x}{a}\right). \tag{II–5}$$

(a) Obtain the wave function at a later time $t = t_0$.

(b) Find the average energy at $t = 0$ and at $t = t_0$.

(c) Obtain the probability that at $t = t_0$, the particle is found in the left quarter of the region, i.e., in $0 \le x \le a/4$.

II–3. Consider a grounded conducting sphere of radius R. A point-like electric dipole of moment p is placed at distance $a > R$ from the center of the sphere. The dipole is oriented along the radial direction. Find the induced charge distribution on the surface of the sphere.

II–4. Assume that the earth and the sun are perfect black bodies.

(a) How much energy does the sun radiate in Watts?

(b) What fraction of the sun's radiation is captured by the earth?

(c) What is the earth's intake of radiation energy from the sun?

(d) Assume that the energy intake from the sun equals the energy radiated by the earth. Then derive the average temperature at the surface of the earth.

Apart from constants included in the table of fundamental constants, other constants you may need are:

- Radius of earth is $R_E = 6380$ km
- Radius of sun is $R_S = 7.0 \times 10^8$ m
- Sun-earth separation is $R_{SE} = 1.5 \times 10^{11}$ m
- Sun's surface temperature is $T_S = 6230$ K

I-1

a)

$$I_{CM} = \int_{-L/2}^{+L/2} m\frac{dx}{L}\,x^2 = \frac{1}{12}mL^2$$

$$I_{\text{Left}} = I_{CM} + m(L/2)^2 = \frac{1}{3}mL^2$$

b) Call it T, then the equations for acceleration and angular acceleration read

$$mg - T = m\frac{L}{2}\alpha$$

$$mg\frac{L}{2} = \frac{1}{3}mL^2\alpha$$

yielding

$$\boxed{T = \frac{1}{4}mg}$$

I-2

Defining

$$S_\pm = S_x \pm iS_y$$

the Hamiltonian can be re-written as

$$H = \frac{A}{2}(S_+^2 + S_-^2) + BS_z^2$$

The Hilbert space is the direct sum of two parts:

$$\frac{1}{2}\otimes\frac{1}{2} = 0 \oplus 1$$

The spin zero part is the singlet state

$$|j=0, m=0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$$

And the spin one part is

$$|j=1, m=-1\rangle = |\downarrow\downarrow\rangle$$

$$|j=1, m=0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$$

$$|j=1, m=+1\rangle = |\uparrow\uparrow\rangle$$

Interestingly, the Hamiltonian respects this decomposition as well

$$H = H_0 \oplus H_1$$

with

$$H_0 = (0) \quad ; \quad H_1 = \begin{pmatrix} B & 0 & 2A \\ 0 & 0 & 0 \\ 2A & 0 & B \end{pmatrix}$$

Therefore, the diagonalization is as follows

$$E = B \pm 2A: \quad |\pm\rangle = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\rangle \pm |\downarrow\downarrow\rangle)$$

$$E = 0: \quad |0_\pm\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle \pm |\downarrow\uparrow\rangle)$$

I-3

The attractive force is given by

$$F = \frac{Q^2}{2Cd} = \frac{CV^2}{2d}$$

While C itself depends on the distance d as $C = K/d$.

a) Doubling the distance, doubles the capacitance. With the battery connected, the potential difference is held constant, and therefore

$$W = \frac{1}{2}KV^2 \int_d^{2d} \frac{dx}{x^2} = \frac{1}{2}KV^2 \frac{1}{2d} \approx \boxed{1.84 \times 10^{-5}\, J}$$

b) Here, the charge is constant

$$W = \frac{Q^2}{2K}\int_d^{2d} dx = \frac{Q^2}{2C} = \frac{1}{2}CV^2 \approx \boxed{3.68 \times 10^{-5}\, J}$$

I-4

Of the

$$dN = ndV \frac{A_0 \cos\theta}{4\pi r^2}$$

particles in a volume element dV, that are aimed at the hole, those that are moving faster than $r/\delta t$ will make it out of the container before a time δt passes, and carry an energy $\frac{1}{2}mv^2$ with themselves. Therefore, if $f(v)$ is the speed probability distribution of the particles, then

$$\delta E = 2\pi n \int_0^\infty dr\, r^2 \int_0^{\pi/2} d\theta\, \sin\theta \frac{A_0 \cos\theta}{4\pi r^2} \int_{r/\delta t}^\infty \frac{1}{2}mv^2 f(v)dv$$

$$= \frac{1}{8}mnA_0 \int_0^\infty dr \int_{r/\delta t}^\infty v^2 f(v)dv = \frac{1}{8}mnA_0\delta t \int_0^\infty dv f(v)v^3$$

$$= \delta t \times nkTA_0\sqrt{\frac{2kT}{\pi m}}$$

II-1

a)

$$L = T - V = \frac{1}{2} m R^2 (\dot{\theta}^2 + \sin^2 \theta \Omega^2) + mgR \cos\theta$$

This is equivalent to a non-rotating frame with potential

$$V_{\text{eff}}(\theta) = -mgR\cos\theta - \frac{1}{2} m R^2 \Omega^2 \sin^2 \theta$$

b, c) Demanding $\dot{\theta} = 0$ to be a solution, follows the equation

$$\sin\theta \left(1 - \frac{R\Omega^2}{g} \cos\theta\right) = 0$$

The equilibria at $\theta \in \{0, \pi\}$ are always present. As for their stability, the one at $\theta = \pi$ is always unstable and the one at $\theta = 0$ is stable if and only if $\Omega^2 \leq Rg$ (note that the condition includes the special case of equality as well, it is a quartic stability at that point). As soon as $\Omega > gR$, two new equilibria appear at $\theta_\pm = \pm \arccos\left(gR/\Omega^2\right)$; they are both stable whenever they exist.

II-2

Let us first re-write the wave function in terms of the energy eigenfunctions

$$\psi_n = \sqrt{\frac{2}{a}} \sin \frac{n\pi x}{a} \quad ; \quad E_n = \frac{n^2 \pi^2 \hbar^2}{2ma^2}$$

$$\psi(x, t = 0) = \frac{1}{\sqrt{17}} \left[4\psi_2(x) + \psi_4(x)\right]$$

a)

$$\psi(x, t_0) = \frac{1}{\sqrt{17}} \left[4\psi_2(x) e^{-iE_2 t_0/\hbar} + \psi_4(x) e^{-iE_4 t_0/\hbar}\right]$$

b)

$$\langle H \rangle = \frac{16E_2 + E_4}{17} = \frac{40\pi^2\hbar^2}{2ma^2} \quad \forall t$$

c)

$$P = \int_0^{a/4} |\psi(x,t_0)|^2 dx = \frac{2}{17a} \int_0^{a/4} dx \left[16 \sin^2 \frac{2\pi x}{a} + \sin^2 \frac{4\pi x}{a} + 8 \sin \frac{2\pi x}{a} \sin \frac{4\pi x}{a} \cos \frac{(E_4 - E_2)t_0}{\hbar}\right]$$

$$= \frac{1}{17} \left[4 + \frac{1}{4} + \frac{16}{3\pi} \cos \frac{(E_4 - E_2)t_0}{\hbar}\right] = \boxed{\frac{1}{4} + \frac{16}{51\pi} \cos \frac{(E_4 - E_2)t_0}{\hbar}}$$

II-3

Let's take the positive direction of the dipole p to be pointing outwards from the center of the sphere. The image charges can be though of as a combination of a dipole and a monopole. The monopole charge of the sphere will be $Q = \frac{pR}{a^2}$ effectively positioned at R^2/a from the center, towards the dipole. There is also a dipole in the opposite direction, at the same spot, pointing towards the center with magnitude $p' = p(R^3/a^3)$. The electric field from a dipole pointing in the z direction at the center of coordinates, in spherical coordinates, is

$$\mathbf{E} = \frac{p}{4\pi\varepsilon_0 r^3}\left(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}\right)$$

Now we focus on finding the surface charge density at an angle θ on the sphere. Here are a few useful definitions

$$r \equiv \left(a^2 + R^2 - 2aR\cos\theta\right)^{1/2}$$

$$r' \equiv \left(R^2 + R^4/a^2 - 2\frac{R^3}{a}\cos\theta\right)^{1/2}$$

$$\alpha = \arcsin\frac{R\sin\theta}{r}$$

$$\beta = \mathrm{sgn}(\theta) \times \arccos\left(\frac{R}{r'}\cos\theta - \frac{R^2}{ar'}\right)$$

Then

$$\sigma = \frac{p}{4\pi a^3}\left\{\frac{Ra}{r'^2}\cos(\theta-\beta) + \frac{R^3}{r'^3}\left[-2\cos\beta\cos(\theta-\beta) + \sin\beta\sin(\beta-\theta)\right] + \frac{a^3}{r^3}\left[2\cos\alpha\cos(\theta+\alpha) - \sin\alpha\sin(\theta+\alpha)\right]\right\}$$

II-4

a)

$$L = 4\pi R_S^2\sigma T_S^4 \approx 5.26 \times 10^{26}$$

b)

$$f = \frac{\pi R_E^2}{4\pi R_{SE}^2} = \approx 4.52 \times 10^{-10}$$

c)

$$fL \approx 2.38 \times 10^{17}$$

d)

$$4\pi R_E^2\sigma T_E^4 = fL \;\;\Rightarrow\;\; T_E = T_S\sqrt{\frac{R_S}{2R_{SE}}} \approx 301\,K$$