PSU Physics PhD Qualifying Exam Solutions
Fall 2014

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August 6, 2023

Qualifying Exam for Ph.D. Candidacy
Department of Physics
October 11, 2014

Part I

Instructions:

- The following problems are intended to probe your understanding of basic physical principles. When answering each question, indicate the principles being applied and any approximations required to arrive at your solution. *If information you need is not given, you may define a variable or make a reasonable physical estimate, as appropriate.* Your solutions will be evaluated based on clarity of physical reasoning, clarity of presentation, and accuracy.
- Please use a new blue book for each question. Remember to write your name and the problem number of the cover of each book.
- We suggest you read all *four* of the problems before beginning to work them. You should reserve time to attempt every problem.

Fundamental constants, conversions, etc.:

Avogadro's number	N_A	$6.022 \times 10^{23}\,\mathrm{mol}^{-1}$
Boltzmann's constant	k_B	$1.381 \times 10^{-23}\,\mathrm{J\,K}^{-1}$
Electron charge magnitude	e	$1.602 \times 10^{-19}\,\mathrm{C}$
Gas constant	R	$8.314\,\mathrm{J\,mol}^{-1}\,\mathrm{K}^{-1}$
Planck's constant	h $\hbar = h/2\pi$	$6.626 \times 10^{-34}\,\mathrm{J\,s}$ $1.055 \times 10^{-34}\,\mathrm{J\,s}$
Speed of light in vacuum	c	$2.998 \times 10^{8}\,\mathrm{m\,s}^{-1}$
Permittivity constant	ϵ_0	$8.854 \times 10^{-12}\,\mathrm{F\,m}^{-1}$
Permeability constant	μ_0	$1.257 \times 10^{-6}\,\mathrm{N\,A}^{-2}$
Gravitational constant	G	$6.674 \times 10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}}$
Standard atmospheric pressure	1 atmosphere	$1.01 \times 10^{5}\,\mathrm{N\,m}^{-2}$
Stefan-Boltzmann constant	σ	$5.67 \times 10^{-8}\,\mathrm{W\,m^{-2}\,K^{-4}}$
Electron rest mass	m_e	$9.109 \times 10^{-31}\,\mathrm{kg} = 0.5110\,\mathrm{MeV}\,c^{-2}$
Proton rest mass	m_p	$1.673 \times 10^{-27}\,\mathrm{kg} = 938.3\,\mathrm{MeV}\,c^{-2}$
Origin of temperature scales		$0\,^\circ\mathrm{C} = 273\,\mathrm{K}$
1 large calorie (as in nutrition) 1 inch		$4.184\,\mathrm{kJ}$ $2.54\,\mathrm{cm}$

Definite integrals:

$$\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}. \tag{I–1}$$

$$\int_0^\infty x^n e^{-x} dx = \Gamma(n+1) = n!. \tag{I–2}$$

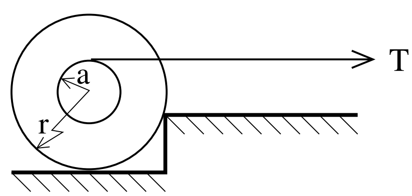
Laplacian in spherical polar coordinates (r, θ, ϕ):

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 f}{\partial \phi^2}. \tag{I–3}$$

Laplacian in cylindrical coordinates (r, θ, z):

$$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}. \tag{I–4}$$

I–1. A pair of wheels of radius r are attached to an axle of radius a. They are pulled over a stepwise obstacle of height h by a rope, as in the diagram. One end of the rope is attached to the axle. The rope is pulled over the top of the axle, and it is horizontal where it leaves the axle; you are to neglect the thickness of the rope. The wheels and axle have mass M and are cylindrically symmetric. The center of mass of the axle-wheel system is midway between the wheels, as is the rope. What minimum tension T needs to be applied to the rope to get the wheel over the obstacle? What minimum static coefficient of friction is needed such that the wheels roll over the obstacle rather than sliding?



I–2. A conductor of mass $m = 160$ g and length $l = 80$ cm is suspended horizontally from two identical massless threads. The system is placed in a vertically oriented homogeneous magnetic field, with $B = 1$ T, and flexible cables (which lie outside the region with the magnetic field) are connected to the conductor so that a current $I = 2$ A flows through it.

- Starting with the Lorentz force, derive the magnetic force on the conductor.
- What is the angle α between the threads and the vertical when the conductor is in its equilibrium position.

I–3. A particle in a one-dimensional harmonic oscillator potential starts out in the quantum mechanical state with wave function

$$\psi(x, t = 0) = A[3\psi_0(x) + 4\psi_1(x)],$$

where

$$\psi_0(x) = \left(\frac{1}{\pi^{1/4} x_0^{1/2}}\right) \exp\left[-\frac{1}{2}\left(\frac{x}{x_0}\right)^2\right]$$

and

$$\psi_1(x) = \left(\frac{2^{1/2} x}{\pi^{1/4} x_0^{3/2}}\right) \exp\left[-\frac{1}{2}\left(\frac{x}{x_0}\right)^2\right]$$

are normalized eigenfunctions of the ground and first excited level, respectively, and $x_0 = \sqrt{\frac{\hbar}{m\omega}}$ is the harmonic oscillator length.

(a) Find A.

(b) What is the form of the state $\psi(x, t)$ and the probability density $|\psi(x, t)|^2$ at time $t > 0$?

(c) Find the expectation value of the position in state $\psi(t)$ at arbitrary time t.

(d) Suppose you measure the energy of this particle. What values could you get and with what probabilities?

I–4. A cylinder 1 m long has a thin, massless, piston clamped in the middle. The cylinder is in a heat bath at $T = 300\,\mathrm{K}$. The left side of the cylinder contains 1 mole of helium gas at 4 atm. The right side of the cylinder has helium gas at a pressure of 1 atm. Let the piston be released to slide to an equilibrium position. Assume that the piston is connected to the outside so that it moves slowly.

(a) What is the final position?

(b) How much heat is transmitted to the bath in the process of reaching equilibrium?

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II–1. Two particles of equal masses $m_1 = m_2$ move on a frictionless horizontal surface in the vicinity of a fixed force center, with potential energies $U_1 = \frac{1}{2}kr_1^2$ and $U_2 = \frac{1}{2}kr_2^2$. In addition, they interact with each other via a potential energy $U_{12} = \frac{1}{2}\alpha k r^2$, where $r = |\vec{r}_1 - \vec{r}_2|$ is the distance between them and α and k are positive constants.

(a) Find the Lagrangian in terms of the center of mass position $\vec{R}$ and the relative position $\vec{r} = \vec{r}_1 - \vec{r}_2$.

(b) Write down and solve the Lagrange equations for the center of mass coordinates $\vec{R} = (X, Y)$ and relative coordinates $\vec{r} = (x, y)$. Describe the motion.

II–2. (a) In a beam of classical non-relativistic particles of mass m, each particle initially moves parallel to the z-axis with velocity v. At a certain plane, perpendicular to the z-axis, each particle experiences a transverse impulse $-b\mathbf{r}_T$, which is proportional to transverse distance $\mathbf{r}_T$ from the z-axis with coefficient $-b$. Show that the particles converge to a point on the z-axis, called a focal point of the system. Find the distance f of the focal point from the plane where the impulse happens. Assume that the z-width of the region of the impulse is negligible compared with f.

[The approximation can be termed a thin lens approximation. It can be shown that this system acting on particles behaves exactly like a lens acting on rays of light, and that f is the focal length.]

(b) A quadrupole magnet is placed in a beam of momentum=200 MeV/c protons traveling in the z-direction near the z axis. The quadrupole magnet is 1 meter long (in the z direction) and, near the beam axis, the magnetic field is approximately given by

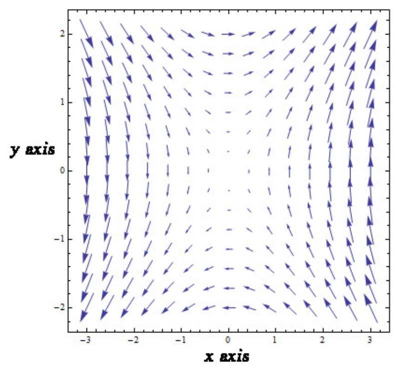
$$\mathbf{B}(x, y, z) = Ky\mathbf{x} + Kx\mathbf{y},$$

where $K = 0.1$ T/m.

(i) Describe in words the effect this magnet has on the beam of protons.

(ii) Determine the focal length of this lens-like system in both the x-z and y-z plane. (Use the non-relativistic and the thin-lens approximations.)

In solving this problem, include a brief assessment of the approximations used.



II–3. Consider a single quantum mechanical particle of mass m in a one-dimensional infinite square well of length ℓ.

(a) Determine the energy eigenvalues and eigenfunctions of the system.

Now consider two (non-interacting) particles of mass m with spin in the same infinite square well. They are in the same spin state.

(b) What is the wavefunction for the ground state and what is the value of the energy for the ground state if the particles are identical bosons?

(c) What is the wavefunction for the first excited state and what is the value of the energy for this state?

(d) Suppose the particles are identical fermions. What is the value of the ground state energy and the wave function corresponding to this state?

(e) What is the value of the energy for the first excited state in the fermion case?

II–4. The vibrational energy levels of a molecule in a gas at temperature T are often described by the harmonic oscillator spectrum, $E_n = \hbar\omega n$, where $n = 0, 1, 2, \ldots$. (We ignore the zero-point energy).

(a) Suppose that the system consists of N molecules whose vibrational levels fit the preceding description. Evaluate the fraction f_n of these molecules which have vibrational quantum number n.

(b) Determine the mean vibrational energy $\langle E \rangle$ of a molecule.

(c) Consider a system consisting of exactly $N = 3$ molecules, which share a total vibrational energy $E_{\text{total}} = 3\hbar\omega$. Evaluate the entropy of this system, assuming that the molecules are distinguishable.

I-1

When on the brink of skipping the obstacle, the normal force from the ground is zero. Writing the torque equilibrium condition around the obstacle corner, we have

$$T(a+r-h) = Mgr \quad \Rightarrow \quad T = \frac{Mgr}{a+r-h}$$

The friction should support this force on the tip of the obstacle

$$\mu \geq \left| \tan\left[\arctan\left(\frac{r+a-h}{r}\right) - \arcsin\left(1-\frac{h}{r}\right)\right] \right|$$

I-2

The magnetic force on a piece of the current carrying conductor is

$$\mathbf{F} = (q_i \mathbf{v}_i) \times \mathbf{B} = \Delta V\, \mathbf{J} \times \mathbf{B} = \Delta l\, \mathbf{I} \times \mathbf{B}$$

Specifically, in our case:

$$\mathbf{F} = 1.6\, N\, \hat{\mathbf{n}}$$

Where $\hat{\mathbf{n}}$ is a horizontal direction normal to the conductor. Incidentally, this is similar to the weight force on the conductor, therefore, the tilt angle is $\alpha \approx \pi/4$.

I-3

a) From normalization: $A = \frac{1}{5}$

b)

$$\rho(x) = \frac{e^{-x^2/x_0^2}}{25\sqrt{\pi}x_0}\left|3 + \frac{4\sqrt{2}x}{x_0}e^{-i\omega t}\right|^2 = \frac{e^{-x^2/x_0^2}}{25\sqrt{\pi}x_0}\left(9 + \frac{32x^2}{x_0^2} + \frac{24\sqrt{2}x}{x_0}\cos\omega t\right)$$

c)

$$\langle\psi(t)|\, x\, |\psi(t)\rangle = \frac{1}{25}\left(9\langle\psi_0|\, x\, |\psi_0\rangle + 16\langle\psi_1|\, x\, |\psi_1\rangle + 24\langle\psi_0|\, x\, |\psi_1\rangle \cos\omega t\right) = \frac{24x_0}{25\sqrt{2}}\cos\omega t$$

d) It will be $\frac{1}{2}\hbar\omega$ with probability 9/25, and $\frac{3}{2}\hbar\omega$ w.p. 16/25.

I-4

a) Since the piston is in the middle, the right hand side has 1/4 mole Hellium gas. In the final position, the volume in each part is proportional to the mole number: the piston will be 80cm from the left end and 20cm from the right end.

b) Assuming that the gasses behave ideally, the change in their energy content is zero (no temperature change). The net external work is also zero, meaning there is no heat exchange.

II-1

a)

$$L = \frac{1}{2}m\left(2\dot{\mathbf{R}}^2 + \frac{1}{2}\dot{\mathbf{r}}^2\right) - \frac{1}{2}k\left[2R^2 + (\frac{1}{2} + \alpha)r^2\right]$$

b)

$$\ddot{\mathbf{R}} + \frac{k}{m}\mathbf{R} = \mathbf{0} \quad\Rightarrow\quad \mathbf{R} = \mathbf{A}\cos(\omega t) + \mathbf{B}\sin(\omega t)$$

$$\ddot{\mathbf{r}} + \frac{k}{m}(2\alpha+1)\mathbf{r} = \mathbf{0} \quad\Rightarrow\quad \mathbf{r} = \mathbf{C}\cos\big(\omega\sqrt{2\alpha+1}t\big) + \mathbf{D}\sin\big(\omega\sqrt{2\alpha+1}t\big)$$

In general, the motion consists of an elliptic orbit moving around another elliptic orbit.

II-2

a)

$$f = v \times \frac{r_T}{br_T/m} = \frac{mv}{b}$$

b) The non-relativistic approximation holds best when $p/mc \ll 1$, in this case: $p/mc \approx 0.21$. The equations of motion are

$$\ddot{x} = -\frac{Ke}{m}\dot{z}x \quad;\quad \ddot{y} = \frac{Ke}{m}\dot{z}y \quad;\quad \ddot{z} = \frac{Ke}{m}(x\dot{x} - y\dot{y})$$

If the velocity $\dot{z}$ does not change much in the magnetic region, then this is similar to the lens problem we had

$$b_x = Ke\dot{z}\frac{\ell}{\dot{z}} \quad\Rightarrow\quad f_x = \frac{mv_z}{Ke\ell} \approx 7m$$

$$b_y = -Ke\dot{z}\frac{\ell}{\dot{z}} \quad\Rightarrow\quad f_y = -\frac{mv_z}{Ke\ell} \approx -7m$$

Hwew ℓ is the length of the magnetic region ($\ell = 1m$). The relative change in the z velocity is easily found as

$$\frac{\Delta v_z}{v_z} = \frac{Ke}{2mv_z}|\Delta(x^2 - y^2)| \lesssim \frac{1}{2}\left(\frac{Ke\ell r}{mv_z}\right)^2 = \frac{1}{2}(\frac{r}{f})^2$$

Therefore, the lens approximation is good when the beam width is much smaller than the focal length of about $7m$.

II-3

a)

$$E_n = \frac{\hbar^2 n^2 \pi^2}{2m\ell^2} \quad;\quad \psi_n(x) = \sqrt{\frac{2}{\ell}}\sin\frac{n\pi x}{\ell}$$

b)

$$|\psi_{g,B}\rangle = |1\rangle \otimes |1\rangle \quad;\quad E_g = \frac{\hbar^2\pi^2}{m\ell^2}$$

c)

$$|\psi_{e,B}\rangle = \frac{1}{\sqrt{2}}(|1\rangle \otimes |2\rangle + |2\rangle \otimes |1\rangle) \quad;\quad E_{e,B} = \frac{5\pi^2\hbar^2}{2m\ell^2}$$

d)

$$|\psi_{g,F}\rangle = \frac{1}{\sqrt{2}}(|1\rangle \otimes |2\rangle - |2\rangle \otimes |1\rangle) \quad ; \quad E_{g,F} = \frac{5\pi^2\hbar^2}{2m\ell^2}$$

e)

$$|\psi_{e,F}\rangle = \frac{1}{\sqrt{2}}(|1\rangle \otimes |3\rangle - |3\rangle \otimes |1\rangle) \quad ; \quad E_{e,F} = \frac{5\pi^2\hbar^2}{m\ell^2}$$

II-4

a)

$$f_n = \frac{e^{-n\beta\hbar\omega}}{1 - e^{-\beta\hbar\omega}}$$

b)

$$\langle E\rangle = \frac{\hbar\omega e^{\beta\hbar\omega}}{(e^{\beta\hbar\omega} - 1)^2}$$

c)

$$\Omega = |\{(3,0,0),(0,3,0),(0,0,3),(2,1,0),(2,0,1),(1,2,0),(1,0,2),(0,2,1),(0,1,2),(1,1,1)\}| = 10$$

$$\Rightarrow \quad S = k_B \log(10)$$