PSU Physics PhD Qualifying Exam Solutions
Spring 2014

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Qualifying Exam for Candidacy
Department of Physics
February 1, 2014

Part I

Instructions:

- The following problems are intended to probe your understanding of basic physical principles. When answering each question, indicate the principles being applied and any approximations required to arrive at your solution. *If information you need is not given, you may define a variable or make a reasonable physical estimate, as appropriate.* Your solutions will be evaluated based on clarity of physical reasoning, clarity of presentation, and accuracy.
- Please use a new blue book for each question. Remember to write your name and the problem number of the cover of each book.
- We suggest you read all *four* of the problems before beginning to work them. You should reserve time to attempt every problem.

Fundamental constants, conversions, etc.:

Avogadro's number	N_A	$6.022 \times 10^{23}\,\mathrm{mol}^{-1}$
Boltzmann's constant	k_B	$1.381 \times 10^{-23}\,\mathrm{J\,K}^{-1}$
Electron charge magnitude	e	$1.602 \times 10^{-19}\,\mathrm{C}$
Gas constant	R	$8.314\,\mathrm{J\,mol}^{-1}\,\mathrm{K}^{-1}$
Planck's constant	h $\hbar = h/2\pi$	$6.626 \times 10^{-34}\,\mathrm{J\,s}$ $1.055 \times 10^{-34}\,\mathrm{J\,s}$
Speed of light in vacuum	c	$2.998 \times 10^{8}\,\mathrm{m\,s}^{-1}$
Permittivity constant	ϵ_0	$8.854 \times 10^{-12}\,\mathrm{F\,m}^{-1}$
Permeability constant	μ_0	$1.257 \times 10^{-6}\,\mathrm{N\,A}^{-2}$
Gravitational constant	G	$6.674 \times 10^{-11}\,\mathrm{m}^3\,\mathrm{kg}^{-1}\,\mathrm{s}^{-2}$
Standard atmospheric pressure	1 atmosphere	$1.01 \times 10^{5}\,\mathrm{N\,m}^{-2}$
Stefan-Boltzmann constant	σ	$5.67 \times 10^{-8}\,\mathrm{W\,m}^{-2}\,\mathrm{K}^{-4}$
Electron rest mass	m_e	$9.109 \times 10^{-31}\,\mathrm{kg} = 0.5110\,\mathrm{MeV}\,c^{-2}$
Proton rest mass	m_p	$1.673 \times 10^{-27}\,\mathrm{kg} = 938.3\,\mathrm{MeV}\,c^{-2}$
Origin of temperature scales		$0\,^\circ\mathrm{C} = 273\,\mathrm{K}$
1 large calorie (as in nutrition) 1 inch		$4.184\,\mathrm{kJ}$ $2.54\,\mathrm{cm}$

Definite integrals:

$$\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}. \tag{I–1}$$

$$\int_0^\infty x^n e^{-x} dx = \Gamma(n+1) = n!. \tag{I–2}$$

Transformation of Lorentz 4-vector (e.g., $(ct, \mathbf{x})$) under boost by velocity v in z direction:

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{1-v^2/c^2}} & 0 & 0 & \frac{v/c}{\sqrt{1-v^2/c^2}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{v/c}{\sqrt{1-v^2/c^2}} & 0 & 0 & \frac{1}{\sqrt{1-v^2/c^2}} \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \tag{I–3}$$

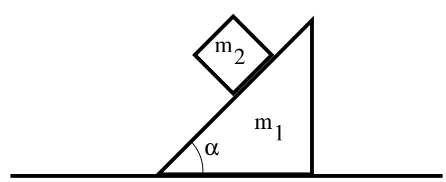
Laplacian in spherical polar coordinates (r, θ, ϕ):

$$\nabla^2 f = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial f}{\partial r} \right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 f}{\partial \phi^2}. \tag{I–4}$$

Laplacian in cylindrical coordinates (r, θ, z):

$$\nabla^2 f = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial f}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}. \tag{I–5}$$

I–1. A block of triangular cross section and of mass m_1 is placed on a horizontal surface. On a diagonal face of the first block is placed another block, of mass m_2. This face is at an angle of α above the horizontal. The coefficients of dynamical friction on the top surface is μ, while the bottom surface is frictionless. Initially the blocks are held stationary, and at one particular time they are released. The conditions are such that there is sliding at both surfaces. Find the acceleration (including direction) of the first block (in terms of the masses, μ and the gravitational acceleration g).



I–2. A thick spherical shell (inner radius a, outer b) is made of dielectric material with a frozen-in polarization

$$\mathbf{P} = \frac{k}{r} \hat{\mathbf{r}}, \tag{I–6}$$

where k is a constant, r is the distance from the center (there is no free charge), and $\hat{\mathbf{r}}$ is the unit vector in the radial direction. Find the electric field everywhere.

I–3. Consider a two-state system with two observables, A and B each taking two values, a_1 and a_2 and b_1 and b_2, respectively. When A takes value a_i the normalized wave function of the system is $|\psi_i\rangle$ while when B takes the value b_i the normalized wave function of the system is $|\phi_i\rangle$. These wave functions are related to each other by

$$|\psi_1\rangle = \frac{3}{5}|\phi_1\rangle + \frac{4}{5}|\phi_2\rangle, \qquad |\psi_2\rangle = \frac{4}{5}|\phi_1\rangle - \frac{3}{5}|\phi_2\rangle. \tag{I–7}$$

(a) The observable A is measured and the value a_1 is obtained. What is the state of the system (immediately) after this measurement?

(b) If B is now measured, what are the possible results, and what are their probabilities?

(c) Right after the measurement of B, A is measured again. What is the probability of getting a_1? (Note that the outcome of the B measurement is not specified.)

I–4. A spherical black body of radius R_1 is maintained at a constant absolute temperature T by internal processes. It is surrounded by a thin spherical and concentric shell of radius R_2, black on both sides. The exterior temperature is T_0.

(a) Find the equilibrium temperature of the outer spherical shell.

(b) Find the ratio between the rate of energy loss in the presence and in the absence of the outer shell.

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II–1. Two protons of equal mass $m_p = 0.938\,\mathrm{GeV}/c^2$ and equal energy $E_0 = 100\,\mathrm{GeV}$ collide elastically. Denote the initial energies by E_{i1} and E_{i2}, and the initial 3-momenta by $\mathbf{p}_{i1}$ and $\mathbf{p}_{i2}$. After the collision, the energies and 3-momenta are E_{f1}, E_{f2}, $\mathbf{p}_{f1}$ and $\mathbf{p}_{f2}$. The energies and momenta are given by

$$E_{i1} = 100\,\mathrm{GeV}, \qquad \mathbf{p}_{i1} = p_0\hat{\mathbf{z}}, \tag{II–6a}$$
$$E_{i2} = 100\,\mathrm{GeV}, \qquad \mathbf{p}_{i2} = -p_0\hat{\mathbf{z}}, \tag{II–6b}$$
$$E_{f1} = 100\,\mathrm{GeV}, \qquad \mathbf{p}_{f1} = p_0(\hat{\mathbf{x}}\sin\theta + \hat{\mathbf{z}}\cos\theta), \tag{II–6c}$$
$$E_{f2} = 100\,\mathrm{GeV}, \qquad \mathbf{p}_{f2} = -p_0(\hat{\mathbf{x}}\sin\theta + \hat{\mathbf{z}}\cos\theta). \tag{II–6d}$$

Assume the scattering angle θ is .01 radians.

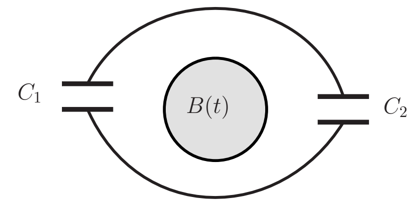
(a) Determine p_0 in units of GeV/c and determine $\gamma = 1/\sqrt{1-\beta^2}$ for the boost from the rest-frame of the proton. Here $\beta = v/c$, with v being the speed of one proton in the center-of-mass reference frame.

(b) The same collision process is now observed in a reference frame where the target proton (2) is at rest before the collision. This is a frame boosted along the z axis from the original frame. What will the momenta of the two protons be after collision? Give x and z components for both final state protons.

(This is the transformation that describes a result from a collider experiment as seen in the frame common for fixed target experiments)

II–2. A planar circuit surrounds a solenoid and consists of two capacitors of capacitances C_1 and C_2 joined together by normal wires. The solenoid crosses the plane of the circuit in a patch of area A, and it produces a time-dependent magnetic field that is changing linearly with time: $B(t) = B_0 + \dot{B}t$; the positive direction is coming up out of the paper. The field is uniform inside the solenoid, and the return path for the flux is well outside the region shown on the picture, and the magnetic field outside the solenoid is to be neglected.

Before the field is applied the capacitors have zero charge. In equilibrium what are the charges Q_1 and Q_2 on the capacitors. Determine the signs.



II–3. A quantum mechanical particle of mass m in one dimension has the following square well potential energy:

$$V(x) = \begin{cases} 0 & \text{if } |x| \le a, \\ V_0 & \text{if } |x| > a. \end{cases} \tag{II–7}$$

Derive an equation whose solution gives the energy eigenvalue(s) for antisymmetric wave functions: $\phi(-x) = -\phi(x)$.

II–4. In 1906, J. B. Perrin started a series of experiments to determine Avogadro's number, for which he was awarded the Nobel Prize in Physics in 1926. In those experiments, he used a microscope to measure the change in concentration of little spherical particles in water with the distance from the bottom of the container. The density of those particles (which he obtained from the resin called gamboge) was $\rho = 1.21 \times 10^3\,\mathrm{kg/m^3}$ and their volume $V = 1.03 \times 10^{-19}\,\mathrm{m^3}$, while the density of water is $\rho_W = 1.00 \times 10^3\,\mathrm{kg/m^3}$. The experiment was done at a temperature $T = 4\,^\circ\mathrm{C}$. Determine the distance from the bottom of the container at which the concentration of those particles halved.

I-1

Writing the second law for the first block, we get

$$m_1 a_1 = N(\sin\alpha - \mu\cos\alpha)$$

But, (best seen while moving with the first block)

$$N = m_2(g\cos\alpha - a_1\sin\alpha)$$

Therefore

$$m_1 a_1 = m_2(g\cos\alpha - a_1\sin\alpha)(\sin\alpha - \mu\cos\alpha)$$

leading to

$$\boxed{a_1 = \frac{m_2 g\cos\alpha\,(\sin\alpha - \mu\cos\alpha)}{m_1 + m_2\sin\alpha\,(\sin\alpha - \mu\cos\alpha)}}$$

I-2

The electric displacement field

$$\boldsymbol{\nabla}\cdot\mathbf{D} \equiv \varepsilon_0\mathbf{E} + \mathbf{P}$$

Spherical symmetry limits this to $\mathbf{D} = C\hat{\mathbf{r}}/r^2$, but there is no charge in the center to cause the singularity, so we have $C = 0$, and hence

$$\boxed{\mathbf{E} = -\frac{1}{\varepsilon_0}\mathbf{P} = \frac{-k\hat{\mathbf{r}}}{\varepsilon_0 r}\times 1[a < r < b]}$$

I-3

a) It is $|\psi_1\rangle$.

b) The result will be b_1 with probability 9/25, and b_2 with probability 16/25.

c) the probability of getting a_1 is

$$P_1 = |\langle\psi_1|\phi_1\rangle|^4 + |\langle\psi_2|\phi_2\rangle|^4 = \frac{81+256}{625} = \frac{337}{625}$$

and therefore

$$P_2 = \frac{288}{625}$$

II-1

a)

$$\gamma = E/m \approx 106.6$$

$$p_0 = \sqrt{(E/c)^2 - m_p^2c^2} \approx 99.9956\,GeV/c$$

b) Defining $E_0 = 100\,GeV$, in the CM frame we have

$$P^\mu_{i1} = (E_0/c, 0, 0, p_0) \quad ; \quad P^\mu_{i2} = (E_0, 0, 0, -p_0)$$

$$P^\mu_{f1} = (E_0/c, p_0 \sin\theta, 0, p_0 \cos\theta) \quad ; \quad P^\mu_{f2} = (E_0/c, -p_0 \sin\theta, 0, -p_0 \cos\theta)$$

The Lorentz transformation to the frame where the second proton is initially at rest, is

$$\Lambda = \begin{pmatrix} \gamma & 0 & 0 & \gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma\beta & 0 & 0 & \gamma \end{pmatrix}$$

with $\beta = p_0 c/E_0$. Giving

$$\bar{P}^\mu_{i1} \approx (21321, 0, 0, 21321)\,GeV/c \quad ; \quad \bar{P}^\mu_{i2} = (0.938, 0, 0, 0)\,GeV/c$$

$$\bar{P}^\mu_{f1} = (21320, 1, 0, 21320)\,GeV/c \quad ; \quad \bar{P}^\mu_{f2} = (1.471, -1, 0, 0.533)\,GeV/c$$

II-2

In equilibrium, there is no current, and therefore no electric field in the wires. Charge conservation reads

$$C_1 V_1 = C_2 V_2$$

while the Faraday's law amounts to

$$V_1 + V_2 = -\dot{B}A$$

These are simultaneously solved as

$$V_1 = \frac{-\dot{B}AC_1}{C_1 + C_2} \quad ; \quad V_2 = \frac{-\dot{B}AC_2}{C_1 + C_2}$$

the polarization is as follows: for C_1, the upper plane is positive and for C_2, the lower plane is positive.

II-3

All these solutions start as

$$\psi(x) = \sin(kx)$$

and then are sewed to

$$\psi(x) = Ae^{-\kappa x}$$

with

$$k = \frac{\sqrt{2mE}}{\hbar} \quad ; \quad \kappa = \frac{\sqrt{2m(V_0 - E)}}{\hbar}$$

The continuity equations are

$$\sin(ka) = Ae^{-\kappa a} \quad ; \quad k\cos(ka) = -\kappa Ae^{-\kappa a}$$

summarised as

$$\boxed{-\kappa = k\cot(ka)}$$

II-4

The concentration goes like

$$\exp\left(-\frac{m_{eff}gh}{k_B T}\right) = \exp\left[-\frac{N_A V(\rho - \rho_w)gh}{RT}\right]$$

Therefore

$$H_{1/2} = \frac{RT\log(2)}{N_A V(\rho - \rho_w)g} \approx 12.5\,\mu m$$