PSU Physics PhD Qualifying Exam Solutions
Fall 2015

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September 4, 2023

Qualifying Exam for Ph.D. Candidacy
Department of Physics
October 17th, 2015

Part I

Instructions:

- The following problems are intended to probe your understanding of basic physical principles. When answering each question, indicate the principles being applied and any approximations required to arrive at your solution. *If information you need is not given, you may define a variable or make a reasonable physical estimate, as appropriate.* Your solutions will be evaluated based on clarity of physical reasoning, clarity of presentation, and accuracy.
- Please use a new blue book for each question. Remember to write your name and the problem number of the cover of each book.
- We suggest you read all *four* of the problems before beginning to work them. You should reserve time to attempt every problem.

Fundamental constants, conversions, etc.:

Avogadro's number	N_A	$6.022 \times 10^{23}\,\mathrm{mol}^{-1}$
Boltzmann's constant	k_B	$1.381 \times 10^{-23}\,\mathrm{J\,K}^{-1}$
Electron charge magnitude	e	$1.602 \times 10^{-19}\,\mathrm{C}$
Gas constant	R	$8.314\,\mathrm{J\,mol}^{-1}\,\mathrm{K}^{-1}$
Planck's constant	h $\hbar = h/2\pi$	$6.626 \times 10^{-34}\,\mathrm{J\,s}$ $1.055 \times 10^{-34}\,\mathrm{J\,s}$
Speed of light in vacuum	c	$2.998 \times 10^{8}\,\mathrm{m\,s}^{-1}$
Permittivity constant	ϵ_0	$8.854 \times 10^{-12}\,\mathrm{F\,m}^{-1}$
Permeability constant	μ_0	$1.257 \times 10^{-6}\,\mathrm{N\,A}^{-2}$
Gravitational constant	G	$6.674 \times 10^{-11}\,\mathrm{m}^3\,\mathrm{kg}^{-1}\,\mathrm{s}^{-2}$
Standard atmospheric pressure	1 atmosphere	$1.01 \times 10^{5}\,\mathrm{N\,m}^{-2}$
Stefan-Boltzmann constant	σ	$5.67 \times 10^{-8}\,\mathrm{W\,m}^{-2}\,\mathrm{K}^{-4}$
Electron rest mass	m_e	$9.109 \times 10^{-31}\,\mathrm{kg} = 0.5110\,\mathrm{MeV}\,c^{-2}$
Proton rest mass	m_p	$1.673 \times 10^{-27}\,\mathrm{kg} = 938.3\,\mathrm{MeV}\,c^{-2}$
Origin of temperature scales		$0\,^\circ\mathrm{C} = 273\,\mathrm{K}$
1 large calorie (as in nutrition) 1 inch		$4.184\,\mathrm{kJ}$ $2.54\,\mathrm{cm}$

Definite integrals:

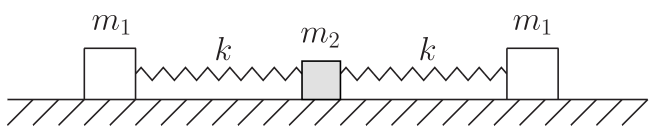
$$\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}. \tag{I–1}$$

$$\int_0^\infty x^n e^{-x} dx = \Gamma(n+1) = n!. \tag{I–2}$$

I–1. A crude approximation of the CO_2 molecule by a classical system is depicted in the figure. It consists of masses $m_1 = 16$ and $m_2 = 12$ (in some units) connected by identical springs as in the figure below; their motion is one-dimensional.

(i) Determine the ratio between the frequencies of the normal vibrational modes.

(ii) Describe the motion of the carbon and oxygen atoms in each vibrational mode.



I–2. A particle of mass m moving in one dimension is confined to the region $0 < x < L$ of zero potential energy by an infinite well potential. In addition, the particle experiences a delta function potential

$$V(x) = \lambda\delta(x - \frac{L}{2}) \ ,$$

where λ is a positive real parameter. Find the equations for the energy eigenvalues E of the particle in terms of the mass m, and the parameters λ and L.

I–3. An ideal monoatomic gas is enclosed in a cylinder of radius a and length L. The cylinder rotates with angular velocity ω about its symmetry axis and the ideal gas is in equilibrium at temperature T in the coordinate system rotating with the cylinder. Assume that the atoms of gas obey classical statistics, each of them has mass m and has no internal degrees of freedom.

(i) What is the Hamiltonian in the rotating coordinate system?

(ii) What is the partition function for the system?

(iii) What is the average number density as a function of r, the distance from the rotation axis?

I–4. Two very long concentric conducting cylindrical shells are located at radius r_1 and r_2 from an axis. The length of the cylinders is L . Between the cylinders is a medium with dielectric constant $\kappa = \varepsilon/\varepsilon_0$. Determine the polarization field in the dielectric medium if the electric potential difference between inner and outer cylinders is V_0 . Determine the expression for the free charge on the inner conductor.

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II–1. A rocket car releases the combustion gas at a speed v with respect to itself and the mass of the combustion gas released per unit time is μ. Assume that the car is initially at rest.

(i) What is the speed of the car after its mass (the joint mass of the structure of the car and the fuel) has decreased n times.

(ii) What is the minimal time required for a car whose structure mass is m to reach the speed of sound v_s.

II–2. (i) Consider a simple one-dimensional quantum mechanical harmonic oscillator of mass m and frequency ω. Starting from the Hamiltonian operator H_0 in terms of the coordinate and momentum operators x and p,

$$H_0 = \frac{P^2}{2m} + V_0 \quad \text{with} \quad V_0 = \frac{m\omega^2 x^2}{2} \ ,$$

express it in terms of creation and annihilation operators. Find its normalized eigenvectors in terms of creation operators and the eigenvalues.

(ii) The anharmonic term $V_1(x) = Kx^4$ is added to the Hamiltonian H_0. Assuming that V_1 is a small perturbation, calculate the first order correction to the energy levels (i.e. the expectation values $\langle\psi_n|V_1(x)|\psi_n\rangle$ where $|\psi_n\rangle$ are the eigenstates of H_0).

II–3. (i) Find the efficiency of a reversible engine operating around the cycle in figure II–1(a). T is the temperature in K and S is the entropy in Joules/K.

(ii) What is the efficiently of the reversible engine operating around the cycle in figure II–1(b)? Is it larger or smaller than the efficiency of an engine operating around the cycle in figure II–1(a)?

II–4. Consider the two circuits shown in the figure II–2. Circuit #1 is a toroidal coil with an inner radius a, an outer radius b, a height c, and wound with N total turns. The toroid is coaxial with one segment of a flat rectangular circuit loop

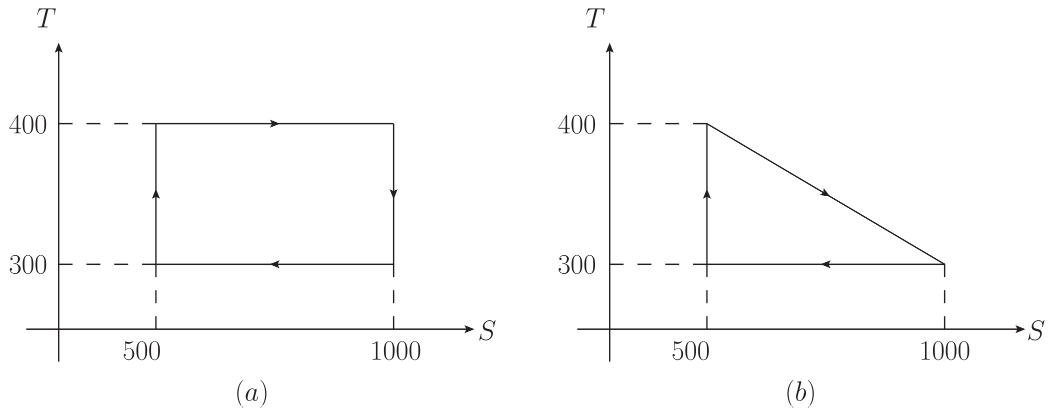

Figure II–1: Thermodynamic cycles

(circuit #2) which lies along the z-axis. The rectangular loop has a height h and a width w. The figure to the right of fig. II–2 shows a cross sectional view in the plane of the rectangular circuit (circuit #2).

(i) Find the mutual inductance M between the two circuits.

(ii) If a time-varying current $I_2 = kt$, flows in the rectangular loop, what is the electro-motive force (EMF) induced in the toroid?

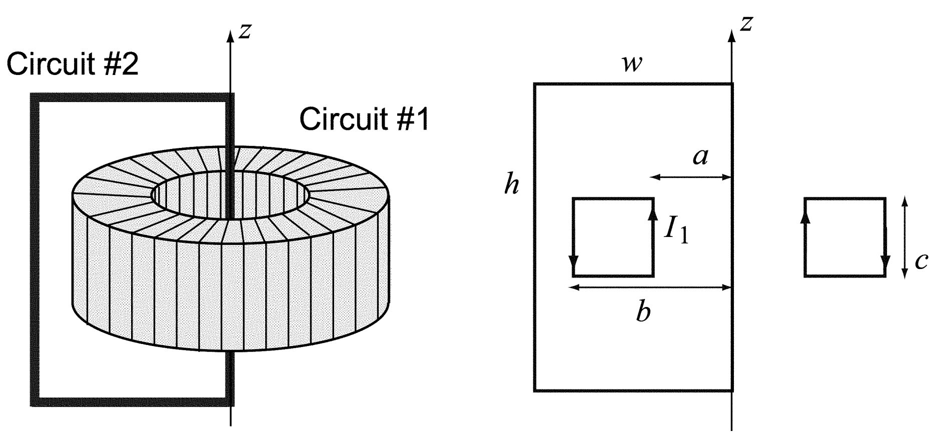

Figure II–2: Circuits

I-1

Let's work in the units where $k = 16$ as well. The equations of motion are

$$\frac{d^2}{dt^2}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} +1 & -1 & 0 \\ -\frac{4}{3} & \frac{8}{3} & -\frac{4}{3} \\ 0 & -1 & +1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

There are 3 modes, the first one is a translational mode: all of the masses move with constant velocity with $\omega = 0$.

$$\mathbf{x} = (1, 1, 1)^T \quad ; \quad \omega = 0$$

Then, there is the oscillation in which the Carbon is still and the Oxygen atoms oscillate

$$\mathbf{x} = (+1, 0, -1)^T \quad ; \quad \omega = 1$$

The next mode has shorter wavelength in that the signs oscillate.

$$\mathbf{x} = (3, -8, 3)^T \quad ; \quad \omega^2 = \frac{11}{3}$$

I-2

The even number states are not ffected by the delta function

$$E_n = \frac{n^2\pi^2\hbar^2}{2mL^2} \quad ; \quad \psi_n = \sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L} \quad n = 2, 4, 6, \cdots$$

As for the odd numbers, the states start with $\sin(kx)$ on each side, then they have to meet the (dis)continuity condition

$$k\cos\left(\frac{kL}{2}\right) = \frac{m\lambda}{\hbar^2}\sin\left(\frac{kL}{2}\right)$$

Defining

$$\chi = kL/2 \quad ; \quad \alpha = \frac{mL\lambda}{2\hbar^2}$$

this equation reads

$$\tan\chi = \frac{\chi}{\alpha}$$

Then the energies are

$$E_n = \frac{2\hbar^2\chi_n^2}{mL^2} \quad ; \quad \psi_n = \sqrt{\frac{2}{L}\left[1 - \frac{\sin(2\chi_n)}{2\chi_n}\right]^{-1}}\sin\left(\frac{2\chi_n x}{L}\right) \quad n = 1, 3, 5, \cdots$$

As λ (and therefore α) grows from zero to infinity, the χ_n move from

$$\chi_n(\alpha = 0) = n\frac{\pi}{2} \quad ; \quad n = 1, 3, 5, \cdots$$

to

$$\chi_n(\alpha = \infty) = n\pi \quad ; \quad n = 1, 3, 5, \cdots$$

I-3

i)

$$H = \frac{1}{2m}(p_r^2 + p_z^2) + \frac{(p_\varphi + \omega m r^2)^2}{2mr^2} - \frac{1}{2}m\omega^2 r^2$$

ii) The single particle partition function is

$$Z = \int \prod_i (\frac{dq_i dp_i}{h}) \exp(-\beta H) = \frac{2\pi L}{h^3} \int dr dp_r dp_z \exp\left[-\beta\left(\frac{p_r^2 + p_z^2}{2m} - \frac{1}{2}m\omega^2 r^2\right)\right] \int dp_\varphi \exp\left[\frac{-\beta(p_\varphi + \omega m r^2)^2}{2mr^2}\right]$$

$$= \frac{2\pi L}{h^3}\left(\frac{2\pi m}{\beta}\right)^{3/2} \int_0^a dr\, r \exp\left(\frac{1}{2}\beta m\omega^2 r^2\right)$$

$$= \frac{2\pi L}{h^3}\left(\frac{2\pi m}{\beta}\right)^{3/2} \frac{1}{\beta m \omega^2}\left(e^{\frac{1}{2}\beta m\omega^2 a^2} - 1\right)$$

iii) The probability distribution in r is proportional to $r\exp\left(\frac{1}{2}\beta m\omega^2 r^2\right)$. The pre-factor $'r'$ is from the circular geometry of uni-radii volume elements. Therefore, the number density is

$$n = \frac{\beta m\omega^2 N e^{\frac{1}{2}\beta m\omega^2 r^2}}{2\pi L(e^{\frac{1}{2}\beta m\omega^2 a^2} - 1)}$$

I-4

In the bulk, $\nabla \cdot \mathbf{E} = 0$ due to homogeneity and $\nabla \cdot \mathbf{D} = 0$. Therefore

$$\mathbf{E} = \frac{V_0}{\log(r_2/r_1)}\frac{\hat{\mathbf{r}}}{r}$$

$$\mathbf{P} = (\varepsilon - \varepsilon_0)\mathbf{E} = \frac{(\varepsilon - \varepsilon_0)V_0}{\log(r_2/r_1)}\frac{\hat{\mathbf{r}}}{r}$$

$$\mathbf{D} = \varepsilon\mathbf{E} = \frac{\varepsilon V_0}{\log(r_2/r_1)}\frac{\hat{\mathbf{r}}}{r}$$

$$\nabla \cdot \mathbf{D} = \rho_f \quad \Rightarrow \quad \sigma_f = \frac{\varepsilon V_0}{r_1 \log(r_2/r_1)}$$

II-1

i) The equation is

$$d[M(t)u(t)] = v\mu dt$$

Therefore

$$u = (n-1)v$$

ii) The necessary n is $v_s/v + 1$ which means the necessary fuel mass is mv_s/v. Finally, the necessary time is

$$T = \frac{mv_s}{v\mu}$$

II-2

i)

$$a \equiv \frac{1}{2}\left(\frac{X}{\sqrt{\hbar/\omega m}} + \frac{iP}{\sqrt{\omega m\hbar}}\right)$$

Then

$$H_0 = \hbar\omega(a^\dagger a + \frac{1}{2})$$

with

$$E_n = \hbar\omega(n + \frac{1}{2}) \quad ; \quad |n\rangle = \frac{a^{\dagger n}}{\sqrt{n!}}\,|0\rangle$$

ii)

$$\delta E_n^{(1)} = K\,\langle n|\,X^4\,|n\rangle = \frac{K\hbar^2}{4\omega^2 m^2}\,\langle n|\,(a+a^\dagger)^4\,|n\rangle$$

$$= \frac{K\hbar^2}{4\omega^2 m^2}\,\langle n|\,aaa^\dagger a^\dagger + aa^\dagger aa^\dagger + aa^\dagger a^\dagger a + a^\dagger a^\dagger aa + a^\dagger aa^\dagger a + a^\dagger aaa^\dagger\,|n\rangle$$

$$= \frac{K\hbar^2}{4\omega^2 m^2}\,\langle n|\,a(1+a^\dagger a)a^\dagger + (1+N)^2 + (1+N)N + a^\dagger(aa^\dagger - 1)a + N^2 + N(N+1)\,|n\rangle$$

$$= \frac{3K\hbar^2}{4\omega^2 m^2}(2n^2 + 2n + 1)$$

II-3

i)

$$\eta_a = 1 - \frac{Q_C}{Q_H} = 1 - \frac{300\Delta S}{400\Delta S} = 25\%$$

ii)

$$\eta_b = 1 - \frac{Q_C}{Q_H} = 1 - \left[\frac{1}{300\times 500}\int_0^1 500d\lambda(400 - 100\lambda)\right]^{-1} = \frac{1}{7} < 25\%$$

Of course, the first cycle was the Carnot cycle.

II-4

i) When a current I is running in the first circuit, we know that from symmetry considerations, inside and outside of the coil we have

$$\mathbf{B} = \frac{\alpha}{s}\hat{\boldsymbol{\varphi}}$$

Inside, this blows up and therefore, $\mathbf{B} = \mathbf{0}$. Inside, we can find the constant using Ampere's law. This gives

$$\mathbf{B} = \frac{\mu_0 NI}{2\pi s}\hat{\boldsymbol{\varphi}}$$

Therefore

$$\Phi_2 = MI = \frac{\mu_0 Nc}{2\pi}\log(b/a)I$$

leading to

$$\boxed{M = \frac{\mu_0 Nc}{2\pi}\log(b/a)}$$

ii)

$$\mathcal{E} = \frac{\mu_0 N c k}{2\pi} \log(b/a)$$