PSU Physics PhD Qualifying Exam Solutions
Spring 2015

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August 9, 2023

Qualifying Exam for Ph.D. Candidacy
Department of Physics
February 7, 2015

Part I

Instructions:

- The following problems are intended to probe your understanding of basic physical principles. When answering each question, indicate the principles being applied and any approximations required to arrive at your solution. *If information you need is not given, you may define a variable or make a reasonable physical estimate, as appropriate.* Your solutions will be evaluated based on clarity of physical reasoning, clarity of presentation, and accuracy.
- Please use a new blue book for each question. Remember to write your name and the problem number of the cover of each book.
- We suggest you read all *four* of the problems before beginning to work them. You should reserve time to attempt every problem.

Fundamental constants, conversions, etc.:

Avogadro's number	N_A	$6.022 \times 10^{23}\,\mathrm{mol}^{-1}$
Boltzmann's constant	k_B	$1.381 \times 10^{-23}\,\mathrm{J\,K}^{-1}$
Electron charge magnitude	e	$1.602 \times 10^{-19}\,\mathrm{C}$
Gas constant	R	$8.314\,\mathrm{J\,mol}^{-1}\,\mathrm{K}^{-1}$
Planck's constant	h $\hbar = h/2\pi$	$6.626 \times 10^{-34}\,\mathrm{J\,s}$ $1.055 \times 10^{-34}\,\mathrm{J\,s}$
Speed of light in vacuum	c	$2.998 \times 10^{8}\,\mathrm{m\,s}^{-1}$
Permittivity constant	ϵ_0	$8.854 \times 10^{-12}\,\mathrm{F\,m}^{-1}$
Permeability constant	μ_0	$1.257 \times 10^{-6}\,\mathrm{N\,A}^{-2}$
Gravitational constant	G	$6.674 \times 10^{-11}\,\mathrm{m}^3\,\mathrm{kg}^{-1}\,\mathrm{s}^{-2}$
Standard atmospheric pressure	1 atmosphere	$1.01 \times 10^{5}\,\mathrm{N\,m}^{-2}$
Stefan-Boltzmann constant	σ	$5.67 \times 10^{-8}\,\mathrm{W\,m}^{-2}\,\mathrm{K}^{-4}$
Electron rest mass	m_e	$9.109 \times 10^{-31}\,\mathrm{kg} = 0.5110\,\mathrm{MeV}\,c^{-2}$
Proton rest mass	m_p	$1.673 \times 10^{-27}\,\mathrm{kg} = 938.3\,\mathrm{MeV}\,c^{-2}$
Origin of temperature scales		$0\,^\circ\mathrm{C} = 273\,\mathrm{K}$
1 large calorie (as in nutrition) 1 inch		4.184 kJ 2.54 cm

Definite integrals:

$$\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}. \tag{I–1}$$

$$\int_0^\infty x^n e^{-x} dx = \Gamma(n+1) = n!. \tag{I–2}$$

Laplacian in spherical polar coordinates (r, θ, ϕ):

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 f}{\partial \phi^2}. \tag{I–3}$$

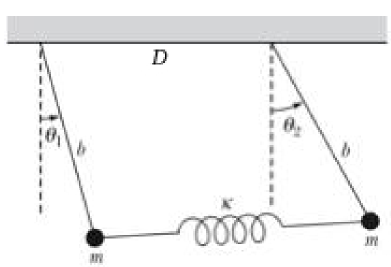
Laplacian in cylindrical coordinates (r, θ, z):

$$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}. \tag{I–4}$$

I–1. Consider two pendula of equal length b and equal masses m connected by a spring of force constant κ. The suspension points are separated by a horizontal distance D as shown. Assume that the spring is of negligible mass and that it is rigid enough that it remains straight and does not buckle or significantly sag. The spring is unstretched in the equilibrium position.

(a) Obtain the Lagrangian for the system and determine the equations of motion.

(b) Determine the normal modes and their angular frequencies for small oscillations from equilibrium.



I–2. A circularly polarized electromagnetic wave in vacuum, moving in the x-direction, has the electric field

$$\vec{E}(\vec{r},t) = E_0 \mathrm{Re}\left(e^{ik(x-ct)}(\vec{e}_y - i\vec{e}_z)\right) .$$

(The vectors $\vec{e}_y$ and $\vec{e}_z$ have unit length and point in the y and z-direction, respectively.)

(a) Compute the magnetic field $\vec{B}(\vec{r},t)$ and the Poynting vector $\vec{S}(\vec{r},t)$ of the wave.

(b) The wave, coming from the negative x direction, encounters an ideal metal plate at $x = 0$. Compute the electric and magnetic fields of the reflected wave.

(c) What is the Poynting vector of the complete electromagnetic field containing the incident wave and the reflected wave?

I–3. A quantum mechanical particle of mass m is constrained to move between two concentric impenetrable shells of radii $r = a$ and $r = b$. There is no other potential. Find the ground state energy and normalized wave-function.

I–4. A system is composed of a large number N of distinguishable atoms at rest, and mutually noninteracting. Each atom has two possible energy states: (i) zero, and (ii) $\varepsilon > 0$. Denote E as the total energy of the system.

(a) What is the average value of E/N when the system is in thermal equilibrium at temperature T?

(b) What is the entropy S when the system is in thermal equilibrium at temperature T?

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Laplacian in cylindrical coordinates (r, θ, z):

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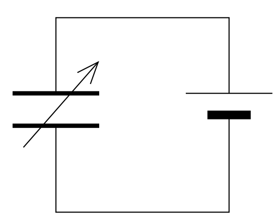
II–1. A spherical shell of negligible thickness and radius R has two holes of radius r_1 and r_2 centered about opposite sides of the same diagonal (cuts created by two parallel planes). The shell is initially at rest. An object with the same mass as the shell explodes in the center. Assuming that the distribution of the very tiny fragments that result from the explosion is spherically symmetric, that those fragments have speed v, and that all the fragments that hit the shell stick to it, determine the velocity of the sphere long time after the explosion.

II–2. (a) Derive a formula for the electrostatic energy in a capacitor of capacitance C when the potential difference across its terminals is V.

(b) A variable capacitor is attached to a source of constant emf E as shown. The capacitance is varied as a function of time such that the current I in the circuit is constant. Initially, the capacitance is C_0. Neglecting internal resistance, compute the power supplied by the emf as a function of time. Compare it with the rate of change of the energy stored in the capacitor, and account for any difference.



II–3. A quantum system is described by a 3-dimensional Hilbert space with orthonormal basis $(|1\rangle, |2\rangle, |3\rangle)$. In this basis, the Hamiltonian $\hat{H}$ is given by the matrix

$$\hat{H} = \epsilon \begin{pmatrix} 4 & 3 & 0 \\ 3 & -4 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

with some constant $\epsilon > 0$ with the units of energy. At time $t = 0$, the system is in the state $|\psi(0)\rangle = 2^{-1/2}(|1\rangle + |3\rangle)$.

(a) Compute $|\psi(t)\rangle$ for $t > 0$.

(b) What is the probability that the system is in state $|2\rangle$ at time t?

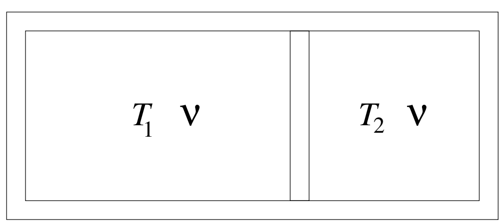
(c) What is the probability that the system is in the ground state at time t?

II–4. A thermally isolated cylindrical container is divided in two parts by a thermally isolating piston that can move freely. Initially, the piston is in its equilibrium position with ν moles of a classical ideal monoatomic gas on each side, with temperatures T_1 and T_2 (see figure). If the piston is removed (assume its volume is negligible compared to the volume of the recipient):

(a) Find the pressure and temperature after the gas comes to equilibrium, the pressure being in terms of the initial pressure P_0.

(b) Compute the change in entropy of the system (after equilibrium is reached).

(c) From your result in (b), discuss whether the entropy decreases, increases, or remains unchanged.



I-1

a)

$$L = \frac{1}{2}mb^2(\dot{\theta}_1^2 + \dot{\theta}_2^2) - \frac{1}{2}mgb(\theta_1^2 + \theta_2^2) - \frac{1}{2}\kappa b^2(\theta_1 - \theta_2)^2$$

leading to

$$\begin{pmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{pmatrix} + \begin{pmatrix} g/b + \kappa/m & -\kappa/m \\ -\kappa/m & g/b + \kappa/m \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = 0$$

b)

$$\omega_\pm^2 = \frac{g}{b} + \frac{\kappa}{m}(1 \pm 1)$$

$$\xi_\pm = \frac{1}{\sqrt{2}} \begin{pmatrix} +1 \\ \mp 1 \end{pmatrix}$$

I-2

a) From $\mathbf{k}.\mathbf{B} = 0$ and $i\mathbf{k} \times \mathbf{B} = i\omega\mathbf{E}$ we find

$$\mathbf{B} = \frac{E_0}{c} \operatorname{Re}\left[e^{ik(x-ct)}(i\hat{\mathbf{y}} + \hat{\mathbf{z}})\right]$$

$$\mathbf{S} = \frac{E_0^2}{\mu_0 c}\hat{\mathbf{x}}$$

b)

$$\mathbf{E}_r = E_0 \operatorname{Re}\left[e^{-ik(x+ct)}(-\hat{\mathbf{y}} + i\hat{\mathbf{z}})\right]$$

$$\mathbf{B}_r = \frac{E_0}{c} \operatorname{Re}\left[e^{-ik(x+ct)}(i\hat{\mathbf{y}} + \hat{\mathbf{z}})\right]$$

c)

$$\mathbf{E}_{\text{tot.}} = 2E_0 \sin(kx)\left[\sin(\omega t)\hat{\mathbf{y}} + \cos(\omega t)\hat{\mathbf{z}}\right]$$

$$\mathbf{B}_{\text{tot.}} = \frac{2E_0}{c} \cos(kx)\left[\sin(\omega t)\hat{\mathbf{y}} + \cos(\omega t)\hat{\mathbf{z}}\right]$$

$$\mathbf{S} = \mathbf{0}$$

I-3

In the ground state, the angular momentum is zero. Therefore $\psi = \psi(r)$ and satisfies

$$\frac{-1}{r^2}\frac{d}{dr}r^2\psi'(r) = k^2\psi(r)$$

with $\psi = \chi(r)/r$ this becomes

$$\chi''(r) = -k^2\chi(r)$$

The ground state corresponds to

$$\chi(r) = A\sin\left[\frac{\pi(r-a)}{b-a}\right]$$

To normalize, we have

$$\frac{1}{A^2} = 4\pi \int_a^b dr\, \sin^2\left[\frac{\pi(r-a)}{b-a}\right] = 2\pi(b-a)$$

Hence

$$\psi = \frac{1}{\sqrt{2\pi(b-a)}r} \sin\left[\frac{\pi(r-a)}{b-a}\right]$$

I-4

a)

$$\frac{\langle E\rangle}{N} = \frac{\varepsilon}{e^{\beta\varepsilon}+1}$$

b)

$$\begin{aligned} S &= Nk_B\left\{\frac{\log\left(1+e^{\beta\varepsilon}\right)}{e^{\beta\varepsilon}+1} + \frac{\log\left(1+e^{-\beta\varepsilon}\right)}{e^{-\beta\varepsilon}+1}\right\} \\ &= Nk_B\left[\log\left(1+e^{\beta\varepsilon}\right) - \frac{\beta\varepsilon}{1+e^{-\beta\varepsilon}}\right] \end{aligned}$$

II-1

The final velocity is

$$\frac{v}{4R^2}|r_1^2 - r_2^2|$$

in the direction of the smaller hole.

II-2

a)

$$U = \frac{1}{2}CV^2$$

b) A KVL shows that the potential across the capacitor is equal to the emf E. Therefore the charge is $Q(t) = C(t)E$. In order to keep the current constant, the capacitance should increase linearly with time

$$C(t) = C_0 + \alpha t$$

In this case, the energy content of the capacitor is

$$U(t) = \frac{1}{2}(C_0 + \alpha t)E^2$$

While the emf power is

$$P = E\dot{Q} = E^2\alpha = 2\dot{U}$$

Half of the energy provided by the battery is wasted during the charging process in the wire. Considering a very small resistance for the wire and re-doing the calculations confirms this.

II-3

a, b, c) The system consists of a state with energy $E_3 = -2\varepsilon$ and two states with energies $E_\pm = \pm 5\varepsilon$:

$$|+\rangle = \frac{1}{\sqrt{10}}(3\,|1\rangle + |2\rangle) \quad ; \quad |-\rangle = \frac{1}{\sqrt{10}}(|1\rangle - 3\,|2\rangle) \quad ; \quad |3\rangle = |3\rangle$$

The initial state is

$$|\psi(0)\rangle = \frac{1}{\sqrt{20}}(3\,|+\rangle + |-\rangle + \sqrt{10}\,|3\rangle)$$

From here, we already have the answer to part (c)

$$P_- = \frac{1}{20}$$

We can evolve the state to find

$$|\psi(t)\rangle = \frac{1}{\sqrt{20}}\left(3e^{-5i\varepsilon t/\hbar}\,|+\rangle + e^{5i\varepsilon t/\hbar}\,|-\rangle + \sqrt{10}e^{2i\varepsilon t/\hbar}\,|3\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left[\left(0.9e^{-5i\varepsilon t/\hbar} + 0.1e^{5i\varepsilon t/\hbar}\right)|1\rangle - 0.6i\sin\left(\frac{5\varepsilon t}{\hbar}\right)|2\rangle + e^{2i\varepsilon t/\hbar}\,|3\rangle\right]$$

The probability of finding the system in state $|2\rangle$ is

$$P_2 = 0.18\sin^2\left(\frac{5\varepsilon t}{\hbar}\right)$$

II-4

a) For mono-atomic ideal gasses, the energy is proportional to temperature. Therefore, the final gas has temperature

$$T_f = \frac{1}{2}(T_1 + T_2)$$

As for the pressure

$$p_f = \frac{2\nu R T_f}{V} = \frac{\nu R(T_1 + T_2)}{\frac{T_1+T_2}{T_1}V_1} = \frac{\nu R T_1}{\nu R T_1}p_0 = p_0$$

b, c)

$$dS = \frac{1}{S}(dE + pdV) = \frac{3}{2}\nu R\frac{dT}{T} + \nu R\frac{dV}{V} \quad \Rightarrow \quad \Delta S = \nu R\left(\frac{3}{2}\Delta\log(T) + \Delta\log(V)\right)$$

For our specific case, this becomes

$$\Delta S = 5\nu R\log\left(\frac{(T_1 + T_2)/2}{\sqrt{T_1 T_2}}\right) > 0$$

The entropy increases (Of course!)