# PSU Physics PhD Qualifying Exam Solutions
# Fall 2016

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# Qualifying Exam for Ph.D. Candidacy
# Department of Physics
# October 1st, 2016

## Part I

Instructions:

- The following problems are intended to probe your understanding of basic physical principles. When answering each question, indicate the principles being applied and any approximations required to arrive at your solution. *If information you need is not given, you may define a variable or make a reasonable physical estimate, as appropriate.* Your solutions will be evaluated based on clarity of physical reasoning, clarity of presentation, and accuracy.
- Please use a new blue book for each question. Remember to write your name and the problem number of the cover of each book.
- We suggest you read all *four* of the problems before beginning to work them. You should reserve time to attempt every problem.

Fundamental constants, conversions, etc.:

| | | |
|---|---|---|
| Avogadro's number | $N_A$ | $6.022 \times 10^{23}\,\mathrm{mol}^{-1}$ |
| Boltzmann's constant | $k_B$ | $1.381 \times 10^{-23}\,\mathrm{J\,K}^{-1}$ |
| Electron charge magnitude | $e$ | $1.602 \times 10^{-19}\,\mathrm{C}$ |
| Gas constant | $R$ | $8.314\,\mathrm{J\,mol}^{-1}\,\mathrm{K}^{-1}$ |
| Planck's constant | $h$<br>$\hbar = h/2\pi$ | $6.626 \times 10^{-34}\,\mathrm{J\,s}$<br>$1.055 \times 10^{-34}\,\mathrm{J\,s}$ |
| Speed of light in vacuum | $c$ | $2.998 \times 10^{8}\,\mathrm{m\,s}^{-1}$ |
| Permittivity constant | $\epsilon_0$ | $8.854 \times 10^{-12}\,\mathrm{F\,m}^{-1}$ |
| Permeability constant | $\mu_0$ | $1.257 \times 10^{-6}\,\mathrm{N\,A}^{-2}$ |
| Gravitational constant | $G$ | $6.674 \times 10^{-11}\,\mathrm{m}^3\,\mathrm{kg}^{-1}\,\mathrm{s}^{-2}$ |
| Standard atmospheric pressure | 1 atmosphere | $1.01 \times 10^{5}\,\mathrm{N\,m}^{-2}$ |
| Stefan-Boltzmann constant | $\sigma$ | $5.67 \times 10^{-8}\,\mathrm{W\,m}^{-2}\,\mathrm{K}^{-4}$ |
| Electron rest mass | $m_e$ | $9.109 \times 10^{-31}\,\mathrm{kg} = 0.5110\,\mathrm{MeV}\,c^{-2}$ |
| Proton rest mass | $m_p$ | $1.673 \times 10^{-27}\,\mathrm{kg} = 938.3\,\mathrm{MeV}\,c^{-2}$ |
| Origin of temperature scales | | $0\,^\circ\mathrm{C} = 273\,\mathrm{K}$ |
| 1 large calorie (as in nutrition)<br>1 inch | | $4.184\,\mathrm{kJ}$<br>$2.54\,\mathrm{cm}$ |

Definite integrals:

$$\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}. \tag{I–1}$$

$$\int_0^\infty x^n e^{-x} dx = \Gamma(n+1) = n!. \tag{I–2}$$

Laplacian in spherical polar coordinates $(r, \theta, \phi)$:

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 f}{\partial \phi^2}. \tag{I–3}$$

Laplacian in cylindrical coordinates $(r, \theta, z)$:

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I–1. A particle with mass $m$ and electric charge $e$ is moving in a two-dimensional plane under the influence of a constant uniform magnetic field pointing in the $z$-direction, $\vec{B} = (0, 0, B)$, and an inverted harmonic oscillator potential,

$$V(x, y) = -\frac{1}{2} m\omega^2 (x^2 + y^2) \ .$$

(i) Write the equations of motion.

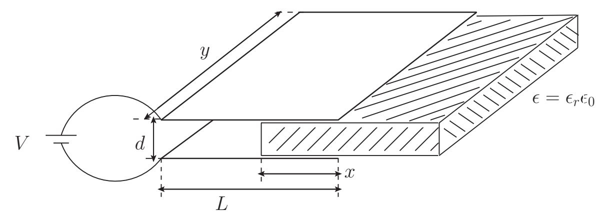
(ii) Find the minimal value of the magnetic field for which the particle exhibits stable motion around the point $x = 0 = y$.

I–2. Consider a particle of mass $m$ moving in a three-dimensional spherical well potential of radius $R$

$$V(|\vec{r}|) = \begin{cases} -V_0 & 0 \le |\vec{r}| \le R \\ 0 & R < |\vec{r}| \end{cases}$$

with $V_0 > 0$. Show that for a well of fixed radius $R$, a bound state exists only if the depth of the well has at least a certain minimum value. Calculate that minimum value.

I–3. Consider a parallel plate capacitor, with rectangular plates of dimensions $L$ and $y$. The distance between the plates is $d$. The potential on the plates is kept constant by the battery with voltage $V$ . The rectangular shaped dielectric with dielectric constant $\varepsilon_r = \varepsilon/\varepsilon_0 > 1$ is partially inserted in between the plates (it is distance $x$ inside the capacitor in the direction $L$, see Fig.I–3). Find the capacitance of this system. What is the infinitesimal change in the energy of the capacitor at constant voltage when dielectric is infinitesimally displaced? What is the magnitude and the direction of the force acting on the dielectric?




I–4. Half a liter of water is warmed up in a 10-inch pan (that is, a wide pan with a radius of $r = 12.7$cm). The pan stands on a stove-top burner which has been set at low heat ($T_0 = 95°$C). Before the heat was turned on, the water had the temperature $T = T_0 - \Delta T$ with $\Delta T = 20°$C. Assume that in this situation heat transfer happens mainly by conduction rather than convection. The vertical temperature profile $u(t, z) = T(t, z) - T_0$ is then described well by the heat equation $\partial u/\partial t = k\partial^2 u/\partial z^2$ with $k = 0.14 \cdot 10^{-6}\text{m}^2/\text{s}$. Notice the similarity with the 1-dimensional time-dependent Schrödinger equation.

(i) Compute the height $h$ of the water column.

(ii) The boundary conditions for the vertical temperature profile are $u(t, 0) = 0$ and $u_z(t, h) = 0$ for all $t$, with $u_z(t, z) = \partial u(t, z)/\partial z$. What is the equilibrium distribution of $u(t, z)$?

(iii) Soon after the heat is turned on, the vertical temperature profile is given by $u(0, z) = -\Delta T \sin(\frac{1}{2}\pi z/h)$ for $0 \le z \le h$. Estimate the time it takes for the temperature to approach equilibrium after $t = 0$.

(iv) What is the minimum amount of heat required to reach equilibrium?

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II–1. A spider is hanging by a silk thread on the mast of a ship in New York Harbor.

(i) Find the orientation and the value of the equilibrium angle the thread makes with the vertical (i.e. the direction of gravity), taking into account the rotation of the Earth. Assume that New York Harbor is at latitude $\theta$ and that the Earth radius is $R_E$.

(ii) Assume that the ship sets sail East with constant velocity $v$ tangential to the surface of the Earth and stays at latitude $\theta$. Find the new orientation and the new value of the equilibrium angle the thread makes with the vertical.

II–2. A proton and an electron are in the combined state

$$\chi = \frac{1}{\sqrt{6}}(2\,|1/2, 1/2\rangle|1/2, 1/2\rangle + |1/2, 1/2\rangle|1/2, -1/2\rangle - |1/2, -1/2\rangle|1/2, 1/2\rangle)$$

(i) What are the probabilities of the possible outcomes of measurements of the $z$-component of the first spin, $\hat{S}_z^{(1)}$?

(ii) What are the probabilities of the possible outcomes of measurements of the $z$-component of the total spin, $\hat{S}_z = \hat{S}_z^{(1)} + \hat{S}_z^{(2)}$?

(iii) What are the eigenvalues of the scalar product $\hat{\vec{S}}^{(1)} \cdot \hat{\vec{S}}^{(2)}$ of the individual spins? What are their multiplicities?

(iv) Is it possible for a pair of electrons to be in the state $\chi$?

II–3. A positively charged ion (with charge $+q_e$) and a neutral atom are separated by a distance $r$. The neutral atom has an isotropic polarizability $\alpha$, such that the atom develops a dipole moment $\mathbf{p} = \alpha \mathbf{E}_l$ in the presence of a local electric $\mathbf{E}_l$. Derive an expression for the force between the ion and the neutral atom. Is the force attractive or repulsive?

II–4. A box contains hydrogen atoms in thermal equilibrium at 300 K. Ignoring perturbative effects, the energy of the hydrogen atom levels (labeled by $n$) is given by the expression: $E_n = -13.6\text{eV}/n^2$.

(i) Considering spin, determine the degeneracy of each level of one atom. Remember that the orbital angular momentum quantum number has values $l = 1, \ldots, n-1$.

(ii) Write the expression for the ratio of the number of atoms in level $n$ ($n$ not too large) to those in the ground state.

(iii) Compute this ratio for $n = 2$. What do you conclude from your result?

(iv) What is the limit of the expression you wrote for the ratio as $n$ becomes very large? Can it exceed 1? If so, under what condition(s)?

(v) Is it realistic that the number of atoms with high $n$ could be greater than the number with low $n$? If your answer is at odds with your result(s) in (iv), identify the main physical reason for the discrepancy.

## I-1

i)

$$\ddot{x} = \omega^2 x + \frac{eB}{m}\dot{y} \quad ; \quad \ddot{y} = -\frac{eB}{m}\dot{x} + \omega^2 y \quad ; \quad \ddot{z} = 0$$

ii) Assuming

$$\begin{pmatrix} x \\ y \end{pmatrix} = e^{st}\begin{pmatrix} X \\ Y \end{pmatrix}$$

The equation becomes

$$\begin{pmatrix} s^2 - \omega^2 & -eBs/m \\ +eBs/m & s^2 - \omega^2 \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix} = 0$$

The eigen-frequencies are

$$s = \sigma_1\sqrt{\omega^2 - (eB/2m)^2} + \sigma_2\frac{ieB}{2m} \quad ; \quad \sigma_{1,2} \in \{\pm 1\}$$

For stability, we need

$$B \geq \frac{2\omega m}{e}$$

## I-2

In general, the states for a spherically symmetric potential are in the form

$$\psi = \frac{\chi_{nl}(r)}{r}Y_{lm}(\theta, \varphi)$$

The ground state ought to have $l = 0$.

$$\chi''_{00}(r) - \frac{2mV(r)}{\hbar^2}\chi_{00}(r) = 0$$

The boundary condition at $r = 0$ and $r = \infty$ set the forms and we find the continuity conditions

$$\sin(kR) = Ae^{-\kappa R} \quad ; \quad k\cos(kR) = -\kappa A e^{-\kappa R}$$

The solutions correspond to

$$kR\cot(kR) = -\kappa R$$

With $x \equiv kR$ and using the definitions for $k$ and $\kappa$ in terms of the energy, we find

$$x\cot(x) = -\sqrt{\frac{2mV_0R^2}{\hbar^2} - x^2}$$

The inverted semi-circle on the RHS will cross the $x\cot(x)$ curve at least once with $\kappa > 0$ if and only if its radius is larger than $\pi/2$.

$$\boxed{V_0 > \frac{\pi^2\hbar^2}{8mR^2}}$$

## I-3

Ignoring the edge effects, the capacitance is the sum of those from two parallel capacitors

$$C(x) = \frac{\varepsilon_0 y}{d}(L - x + \varepsilon_r x)$$

The energy is

$$E(x) = \frac{1}{2}CV^2 = \frac{\varepsilon_0 y}{2d}(L - x + \varepsilon_r x)V^2$$

$$\frac{dE}{dx} = \frac{\varepsilon_0 y}{2d}(\varepsilon_r - 1)V^2$$

The work done by the battery during an infinitesimal move $dx$ is

$$dW_B = VdQ = V^2 dC = \frac{\varepsilon_0 y}{d}(\varepsilon_r - 1)V^2 dx$$

From this, we find the work done by a hand/handle keeping the slab from accelerating is negative the change in energy. Therefore

$$F = \frac{\varepsilon_0 y}{2d}(\varepsilon_r - 1)V^2$$

attractive.

## I-4

Let $V$ denote the volume of the water and $r$ the radius of the pan.

i)

$$h = \frac{V}{\pi r^2} \approx 9.87\ mm$$

ii) In equilibrium: $\partial^2 u/\partial z^2 = 0$ therefore

$$u_{eq.}(\infty, z) = 0$$

iii) Separating the variables leads to the solution basis

$$\sin\left[\pi(n + 1/2)z/h\right] \exp\left[-k\pi^2(n + 1/2)^2 t/h^2\right] \quad ; \quad n = 0, 1, 2, 3, \cdots$$

iv) The temperature exponentially approaches equilibrium with time constant

$$\tau_0 = \frac{4h^2}{\pi^2 k}$$

v)

$$Q = \rho V c \Delta T$$

## II-1

a, b) Assuming the ship is sailing with velocity $v$ eastward, the acceleration (including the gravitational pull) is

$$\mathbf{a} = -\left(\Omega + \frac{v}{R\cos\theta}\right)^2 R\cos\theta\left(\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\boldsymbol{\theta}}\right) + g\mathbf{r}$$

where $\Omega$ is the angular frequency of 1 revolution per day. Therefore, the tilt angle is towards the south and in magnitude, it is

$$\arctan\left\{\frac{(\Omega + v/R\cos\theta)^2 R\cos\theta\,\sin\theta}{g - (\Omega\cos\theta + v/R)^2 R}\right\} \approx \frac{R}{g}(\Omega + \frac{v}{R\cos\theta})^2\cos\theta\,\sin\theta$$

## II-2

Let's first write the state in a shorter notation

$$|\chi\rangle = \frac{1}{\sqrt{6}}\left(2\,|00\rangle + |01\rangle - |10\rangle\right)$$

i)

$$S_z^{(1)} = \begin{cases} +1/2 & \text{w.p.} \quad 5/6 \\ -1/2 & \text{w.p.} \quad 1/6 \end{cases}$$

ii) $S_z^{(1+2)}$ has all of the terms in $|\chi\rangle$ as its eigenstates. Therefore

$$S_z^{(1+2)} = \begin{cases} +1 & \text{w.p.} \quad 2/3 \\ 0 & \text{w.p.} \quad 1/3 \end{cases}$$

iii)

$$\mathbf{S}^{(1)}.\mathbf{S}^{(2)} = \frac{1}{2}\left(|\mathbf{S}^{(1+2)}|^2 - |\mathbf{S}^{(1)}|^2 - |\mathbf{S}^{(2)}|^2\right) = \frac{1}{2}\left(-\frac{3}{2} + |\mathbf{S}^{(1+2)}|^2\right)$$

$$= \frac{-3}{4} + \frac{1}{2}j(j+1) = \begin{cases} -\frac{3}{4} & \text{w. mult.} \quad 1 \\ +\frac{1}{4} & \text{w. mult.} \quad 3 \end{cases}$$

iv) No, because it's not anti-symmetric (the first term is not).

## II-3

$$\mathbf{F} = \alpha\mathbf{E}.\boldsymbol{\nabla}\mathbf{E} = \alpha\frac{q_e}{4\pi\varepsilon_0 r^2}\hat{\mathbf{r}}.\boldsymbol{\nabla}\frac{q_e}{4\pi\varepsilon_0 r^2}\hat{\mathbf{r}} = \frac{-\alpha q_e^2}{8\pi^2\varepsilon_0^2 r^5}\hat{\mathbf{r}}$$

## II-4

i)

$$\Omega_n = \sum_{l=0}^{n} 2 \times (2l+1) = 2n^2$$

ii)

$$n^2 \exp\left\{\frac{-13.6\,eV}{k_B T}(1-\frac{1}{n})\right\} \approx n^2 \exp\left[-526(1-\frac{1}{n})\right]$$

iii)

$$\frac{N_2}{N_1} \approx e^{-394} \ll 1$$

The conclusion is that most atoms are in their ground state.

iv) It will always go to infinity and passes unity when $n$ exceeds about $\exp(|E_1|/2k_BT \approx 263) \gg 1$.

v) It is at odds, for any positive temperature this predicts that no electron should be bounded to its nucleus. However, the paradox is solved when we realise that the size of orbitals also grows like $a_n = n^2 a_0$. Therefore, the size of these extremely high $n$ orbitals is so vast that they can not fit in the room/planet/universe as feasible solutions to the Schrödinger equation.