PSU Physics PhD Qualifying Exam Solutions
Fall 2017

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Qualifying Exam for Ph.D. Candidacy
Department of Physics
October 7th, 2017
Part I

Instructions:

- The following problems are intended to probe your understanding of basic physical principles. When answering each question, indicate the principles being applied and any approximations required to arrive at your solution. *If information you need is not given, you may define a variable or make a reasonable physical estimate, as appropriate.* Your solutions will be evaluated based on clarity of physical reasoning, clarity of presentation, and accuracy.
- Please use a new blue book for each question. Remember to write your name and the problem number of the cover of each book.
- We suggest you read all *four* of the problems before beginning to work them. You should reserve time to attempt every problem.

Fundamental constants, conversions, etc.:

Avogadro's number	N_A	$6.022 \times 10^{23}\,\mathrm{mol^{-1}}$
Boltzmann's constant	k_B	$1.381 \times 10^{-23}\,\mathrm{J\,K^{-1}}$
Electron charge magnitude	e	$1.602 \times 10^{-19}\,\mathrm{C}$
Gas constant	R	$8.314\,\mathrm{J\,mol^{-1}\,K^{-1}}$
Planck's constant	h $\hbar = h/2\pi$	$6.626 \times 10^{-34}\,\mathrm{J\,s}$ $1.055 \times 10^{-34}\,\mathrm{J\,s}$
Speed of light in vacuum	c	$2.998 \times 10^{8}\,\mathrm{m\,s^{-1}}$
Permittivity constant	ϵ_0	$8.854 \times 10^{-12}\,\mathrm{F\,m^{-1}}$
Permeability constant	μ_0	$1.257 \times 10^{-6}\,\mathrm{N\,A^{-2}}$
Gravitational constant	G	$6.674 \times 10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}}$
Standard atmospheric pressure	1 atmosphere	$1.01 \times 10^{5}\,\mathrm{N\,m^{-2}}$
Stefan-Boltzmann constant	σ	$5.67 \times 10^{-8}\,\mathrm{W\,m^{-2}\,K^{-4}}$
Electron rest mass	m_e	$9.109 \times 10^{-31}\,\mathrm{kg} = 0.5110\,\mathrm{MeV}\,c^{-2}$
Proton rest mass	m_p	$1.673 \times 10^{-27}\,\mathrm{kg} = 938.3\,\mathrm{MeV}\,c^{-2}$
Origin of temperature scales		$0\,^\circ\mathrm{C} = 273\,\mathrm{K}$
1 large calorie (as in nutrition) 1 inch		$4.184\,\mathrm{kJ}$ $2.54\,\mathrm{cm}$

Definite integrals:

$$\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}. \tag{I–1}$$

$$\int_0^\infty x^n e^{-x} dx = \Gamma(n+1) = n!. \tag{I–2}$$

$$\int_0^\infty \frac{1}{(x^2+a^2)^n} dx = \frac{1}{2a^{2n-1}} \frac{\Gamma(n-\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(n)} \tag{I–3}$$

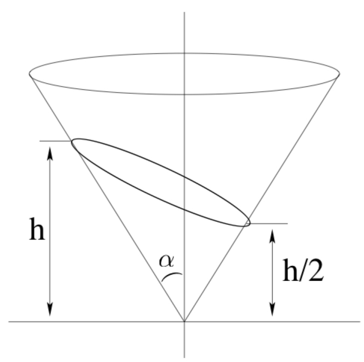
Indefinite integrals:

$$\int \frac{x}{(x^2+a^2)^n} dx = \frac{1}{2(1-n)} \frac{1}{(x^2+a^2)^{n-1}} + c \quad \text{for } n \neq 0, 1 \tag{I–4}$$

I–1. A small object of mass m is moving under the influence of gravity without friction inside a conical surface whose symmetry axis is vertical (see figure). The half-angle at the tip of the cone is α. Gravity acts parallel to the symmetry axis of the cone. Initially, the object was at height h and its velocity was directed horizontally. In its subsequent motion the object descends to a height $h/2$ and then starts climbing back.

a) Write the equations of motion

b) Find the speed of the object at the highest v_{upper} and lowest v_{lower} point of its trajectory



I–2. Consider a particle of mass m in one dimension, subject to a double well delta-function potential

$$V(x) = -g\delta(x-a) - g\delta(x+a) \ .$$

This potential supports at least one bound state for all values of a. For what values of a does this potential support at least two bound states?

I–3. Two parallel conducting plates, P_1 and P_2, have area A and mass M. They are separated by distance d and the plates are perpendicular to the $\hat{z}$ axis. The plate P_1 is held at ground potential and the plate P_2 is held at electric potential V_{P_2} relative to the ground with the use of a battery with internal resistance R.

The plates are large enough or d is small enough so we can assume that the electric field does not depend upon the coordinates x and y spanning the area covered by the plates. In this problem we will look at what happens when we suddenly change the separation of the plates.

a) Determine the capacitance between the plates for separation d at time τ_1 (just before the separation is changed).

Now change the separation from d to $2d$ during the time interval from τ_1 to τ_2. Assume that the change is very rapid, so that no significant charge is provided from the battery between times τ_1 and τ_2.

b) What is the instantaneous voltage across the plates and the instantaneous current flowing into the battery at time τ_2.

c) Find an expression for the voltage across the plates as a function of time for times greater than τ_2.

d) How much heat is dissipated in the internal resistor of the battery between τ_2 and a much later time τ_3, as a function of V_{P_2}, A and d?

I–4. Consider a heated sheet of aluminum of large area A and thickness L along the $\hat{x}$ axis. The heat flow flux $\vec{K}$, defined as the vector power per area of the heat flow, is proportional to the gradient of the temperature,

$$\vec{K} = K_x\hat{x} + K_y\hat{y} + K_z\hat{z} = -\lambda\vec{\nabla}T \ .$$

The heat equation for the temperature $T(x, y, z, t)$ is similar to the Schrödinger's equation:

$$\nabla^2 T = \frac{1}{\alpha}\frac{dT}{dt} \ .$$

Assume that α and λ are constants. We will consider solutions of the heat equation that determine the temperature over the x and t coordinates, $T(x, t)$, where the boundary conditions will be $T(0, t) = 0 = T(L, t)$. (Here "0" stands for room temperature).

a) At $t = 0$ the sheet has an initial temperature distribution

$$T(x, 0) = T_0\left(\sin\left(\frac{\pi}{L}x\right) + \frac{1}{2}\sin\left(\frac{2\pi}{L}x\right)\right) \ ,$$

with T_0 a positive temperature. Evaluate the heat flux K_x emerging from the front and the back surfaces of the sheet ($x = 0$ and $x = L$) at time $t = 0$ in terms of the constants introduced.

b) Separating variables x and t and applying boundary conditions at the surfaces, find the set of separated solutions to the heat equation ($T(x, t) \to Q_n(x)W_n(t)$). Each index n corresponds to a different exponential cooling rate. The general solution would be a superposition of these solutions, with amplitudes A_n, $T(x, t) = \sum_n A_n Q_n(x)W_n(t)$.

c) Determine $T(x, t)$, including the time dependence of the temperature distribution, given the initial conditions from part a).

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Indefinite integrals:

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II–1. A highly relativistic proton with charge 1 and mass $m_p = 0.938 GeV/c^2$ has initial momentum in the $\hat{z}$ direction $\vec{P}_{\text{proton}} = 100 GeV/c\, \hat{z}$. This proton collides elastically with a gold nucleus at rest with an impact parameter $100 fm$. The gold nucleus has been stripped of all electrons and has atomic number $Z = 79$ and atomic weight $197 AMU$. You may assume that the gold nucleus has a radius that is negligible.

a) Integrate $\frac{d\vec{P}_{\text{proton}}}{dt}$ to find the total change in momentum $\Delta\vec{P}_{\text{proton}}$ of the proton, approximating its trajectory with a straight line trajectory at nearly the speed of light through the fixed Coulomb field of the nucleus. Assume that the recoil of the nucleus is negligible.

b) What is the deflection angle of the proton from this scattering process?

II–2. A system of three distinguishable spin-1/2 particles, whose spin operators are $\vec{S}_1$, $\vec{S}_2$ and $\vec{S}_3$, are governed by the Hamiltonian

$$H = \frac{A}{\hbar^2} \vec{S}_1 \cdot \vec{S}_2 + \frac{B}{\hbar^2} (\vec{S}_1 + \vec{S}_2) \cdot \vec{S}_3 \ .$$

Find the energy levels of the system and their degeneracies.

II–3. Two long, straight copper pipes, each of radius R, are held a distance $2d$ apart; we assume that $d > R$. One pipe is held at potential V_0 and the other at potential $-V_0$. Using image charges, find the potential everywhere.

II–4. The idealized Diesel engine cycle consists of four processes. Ideal gas (air) undergoes: (i) an isentropic compression from volume V_1 to volume V_2, (ii) an isobaric heating in which the volume expands to V_3, (iii) an isentropic expansion to volume V_1, and (iv) an isochoric cooling to the initial temperature. Let $r_c = V_2/V_1$ be the compression ratio, $r_e = V_3/V_1$ be the expansion ratio, $\gamma = C_P/C_V$ be the ratio of specific heats of air, and P_2 and V_2 be the pressure and volume, respectively, at the end of process (i).

a) Sketch the P-V diagram for this cycle.

b) Compute the work done by the ideal gas (air) in each process.

c) Compute the amount of heat which is put in the system and the amount that goes out.

d) Compute the efficiency of the idealized Diesel engine.

e) In what limit the efficiency of the idealized Diesel engine becomes the ideal thermodynamic efficiency?

Your results must be written in terms of r_c, r_e, γ, P_2 and V_2.

I - 1

Let the tilt angle and the major semi-axis be β and a respectively. We have

$$2a\sin\beta = \frac{h}{2} \quad ; \quad 2a\cos\beta = \frac{3h}{2}\tan\alpha$$

solved as

$$a = \frac{h}{4}\sqrt{1+9\tan^2\alpha} \quad ; \quad \beta = \arcsin\frac{1}{\sqrt{1+9\tan^2\alpha}}$$

The center of the ellipse is a distance $X_0 = \frac{h}{4}\tan\alpha$ to the left and at a height $z = 3h/4$. At that height, the radius of the cone is $R_0 = \frac{3}{4}h\tan\alpha$. This means that at the center, the minor semi-axis b is found by

$$b^2 + X_0^2 = R_0^2 \quad \Rightarrow \quad b = \frac{h}{\sqrt{2}}\tan\alpha$$

a) The ellipse is parametrized by one angle coordinate as

$$x = a\cos\varphi \;\; ; \;\; y = b\sin\varphi$$

The potential is $V = mga\cos\varphi\sin\beta = \frac{1}{4}mgh\cos\varphi$. This leads to

$$\begin{aligned} L = T - V &= \frac{1}{2}m\left[a^2\sin^2\varphi + b^2\cos^2\varphi\right]\dot{\varphi}^2 - \frac{1}{4}mgh\cos\varphi \\ &= \frac{mh^2\dot{\varphi}^2}{32}\left(8\tan^2\alpha + \frac{\sin^2\varphi}{\cos^2\alpha}\right) - \frac{1}{4}mgh\cos\varphi \end{aligned}$$

From this, the Euler-Lagrange equation of motion is

$$\frac{d}{dt}\left[\frac{h}{2}\left(8\tan^2\alpha + \frac{\sin^2\varphi}{\cos^2\alpha}\right)\dot{\varphi}\right] = g\sin\varphi$$

b) The velocities depend on the initial conditions, from the conservation of energy, we just know that

$$v_{\text{lower}}^2 = v_{\text{upper}}^2 + gh$$

I - 2

Since the potential is even, the eigenstates will be either even or odd. ($[\mathcal{P}, H] = 0$) The first excited state will therefore have a single zero at $x = 0$. We want the energy to be negative and therefore, in the region $0 \le x \le a$ we have

$$\psi(x) = \sinh(\kappa x)$$

Then, in order to be renormalizable, the wave function should break into

$$\psi(x \ge a) = \sinh(\kappa a)e^{\kappa a}e^{-\kappa x}$$

The change in derivative is

$$\frac{-2mg}{\hbar^2}\sinh(\kappa a) = \int_{a^-}^{a^+}\psi''(x)dx = \Delta\psi' = -\kappa\left[e^{\kappa a}\sinh(\kappa a) + \cosh(\kappa a)\right]$$

We can find the minimum g that allows this equation to have a solution via graphical methods; but we already know that the threshold happens when the state is 'barely bound' and therefore we set $\kappa = 0$ and read g_c as

$$\boxed{g_c = \frac{\hbar^2}{2ma}}$$

I - 3

a)

$$\boxed{\frac{\varepsilon_0 A}{d}}$$

b) The electric field does not change so long as the charges don't change (σ/ε_0) and therefore the potential doubles.

$$\boxed{V \to 2V_{P_2}}$$

c)

$$\boxed{V_{P_2}\left(1 + e^{-2dt/R\varepsilon_0 A}\right)}$$

where R is the resistance of the current (internal of the battery, wirings, etc.)

d)

$$W = \int_0^\infty RI^2\,dt = \int_0^\infty dt\,R(C_2\dot{V})^2 = \frac{\varepsilon_0 A}{4d}V_{P_2}^2$$

I - 4

a)

$$K_x(x=0,t=0) = -\lambda\frac{\partial T}{\partial x}|_{x=0,t=0} = -\frac{2\pi\lambda T_0}{L}$$
$$K_x(x=L,t=0) = 0$$

b) For all positive integers $n > 0$:

$$Q_n(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right) \quad ; \quad W_n(t) = \exp\left[-\alpha(\frac{n\pi}{L})^2 t\right]$$

c)

$$T(x,t) = T_0\left[\exp\left(-\frac{\pi^2\alpha t}{L^2}\right)\sin\left(\frac{\pi x}{L}\right) + \frac{1}{2}\exp\left(-\frac{4\pi^2\alpha t}{L^2}\right)\sin\left(\frac{2\pi x}{L}\right)\right]$$

II - 1

a)

$$\Delta\mathbf{P} = \int_{-\infty}^{+\infty}\mathbf{F}\,dt = \int_{-\infty}^{+\infty}dt\,\frac{Ze^2}{4\pi\varepsilon_0}\frac{\mathbf{b}}{r^3(t)} \approx \frac{Ze^2\mathbf{b}}{4\pi\varepsilon_0 c}\int\frac{dx}{(x^2+b^2)^{3/2}} = \frac{Ze^2\hat{\mathbf{b}}}{2\pi\varepsilon_0 cb} \approx 2.275\,MeV/c$$

b)

$$\psi \approx |\frac{\Delta\mathbf{P}}{\mathbf{P}}| \approx 2.275\times 10^{-5}\,\mathrm{rad}$$

II - 2

Let us first re-write the Hamiltonian as

$$
\begin{aligned}
H &= \frac{-A}{2\hbar^2}\left[(\mathbf{S}_1+\mathbf{S}_2)^2 - S_1^2 - S_2^2\right] - \frac{B}{2\hbar^2}\left[(\mathbf{S}_1+\mathbf{S}_2+\mathbf{S}_3)^2 - (\mathbf{S}_1+\mathbf{S}_2)^2 - S_3^2\right] \\
&= \frac{3(2A+B)}{8} - \frac{A+B}{2\hbar^2}(\mathbf{S}_1+\mathbf{S}_2)^2 - \frac{B}{2\hbar^2}(\mathbf{S}_1+\mathbf{S}_2+\mathbf{S}_3)^2
\end{aligned}
$$

Now we just need to know the total angular momenta numbers for $(\mathbf{S}_1+\mathbf{S}_2)$ and $(\mathbf{S}_1+\mathbf{S}_2+\mathbf{S}_3)$.

$$
E = \begin{cases}
\frac{3A}{4} & 0 \oplus \frac{1}{2} = \frac{1}{2} \ \boxed{g=2} \\
-\frac{A+4B}{4} & 1 \oplus \frac{1}{2} = \frac{1}{2} \ \boxed{g=2} \\
-\frac{A+10B}{4} & 1 \oplus \frac{1}{2} = \frac{3}{2} \ \boxed{g=4}
\end{cases}
$$

II - 3

These cylinders are the equipotential surfaces from linear charge distributions $\pm\lambda$ a distance $2a$ apart where

$$
\lambda = \frac{2\pi\varepsilon_0 V_0}{\cosh^{-1}(d/R)} \quad ; \quad a = \sqrt{d^2 - R^2}
$$

Leading to the potential field

$$
\phi = \frac{V_0}{2\cosh^{-1}(d/R)} \log\left(\frac{(x-a)^2+y^2}{(x+a)^2+y^2}\right)
$$

II - 4