PSU Physics PhD Qualifying Exam Solutions
Spring 2017

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Qualifying Exam for Ph.D. Candidacy
Department of Physics
February 4th, 2017

Part I

Instructions:

- The following problems are intended to probe your understanding of basic physical principles. When answering each question, indicate the principles being applied and any approximations required to arrive at your solution. *If information you need is not given, you may define a variable or make a reasonable physical estimate, as appropriate.* Your solutions will be evaluated based on clarity of physical reasoning, clarity of presentation, and accuracy.
- Please use a new blue book for each question. Remember to write your name and the problem number of the cover of each book.
- We suggest you read all *four* of the problems before beginning to work them. You should reserve time to attempt every problem.

Fundamental constants, conversions, etc.:

Avogadro's number	N_A	$6.022 \times 10^{23}\,\mathrm{mol^{-1}}$
Boltzmann's constant	k_B	$1.381 \times 10^{-23}\,\mathrm{J\,K^{-1}}$
Electron charge magnitude	e	$1.602 \times 10^{-19}\,\mathrm{C}$
Gas constant	R	$8.314\,\mathrm{J\,mol^{-1}\,K^{-1}}$
Planck's constant	h $\hbar = h/2\pi$	$6.626 \times 10^{-34}\,\mathrm{J\,s}$ $1.055 \times 10^{-34}\,\mathrm{J\,s}$
Speed of light in vacuum	c	$2.998 \times 10^{8}\,\mathrm{m\,s^{-1}}$
Permittivity constant	ϵ_0	$8.854 \times 10^{-12}\,\mathrm{F\,m^{-1}}$
Permeability constant	μ_0	$1.257 \times 10^{-6}\,\mathrm{N\,A^{-2}}$
Gravitational constant	G	$6.674 \times 10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}}$
Standard atmospheric pressure	1 atmosphere	$1.01 \times 10^{5}\,\mathrm{N\,m^{-2}}$
Stefan-Boltzmann constant	σ	$5.67 \times 10^{-8}\,\mathrm{W\,m^{-2}\,K^{-4}}$
Electron rest mass	m_e	$9.109 \times 10^{-31}\,\mathrm{kg} = 0.5110\,\mathrm{MeV}\,c^{-2}$
Proton rest mass	m_p	$1.673 \times 10^{-27}\,\mathrm{kg} = 938.3\,\mathrm{MeV}\,c^{-2}$
Origin of temperature scales		$0\,^\circ\mathrm{C} = 273\,\mathrm{K}$
1 large calorie (as in nutrition) 1 inch		$4.184\,\mathrm{kJ}$ $2.54\,\mathrm{cm}$

Definite integrals:

$$\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}. \tag{I–1}$$

$$\int_0^\infty x^n e^{-x} dx = \Gamma(n+1) = n!. \tag{I–2}$$

Indefinite integrals:

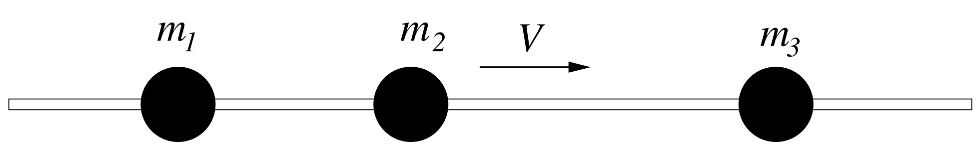
$$\int \frac{x}{(x^2+a^2)^n} dx = \frac{1}{2(1-n)} \frac{1}{(x^2+a^2)^{n-1}} n + c \quad \text{for } n \neq 0, 1 \tag{I–3}$$

I–1. Three spheres of masses m_1, m_2, and m_3 are placed on a thin rod as shown in the figure (they can slide without friction). Initially, m_1 and m_3 are at rest and m_2 has speed V moving toward m_3.

i) State the condition(s) required for m_2 to collide with m_1. Assuming it is (they are) met, determine the speeds of m_1 and m_3 after their first collision with m_2.

ii) Assuming that $m_1 \gg m_2$ and $m_3 \gg m_2$, determine the asymptotic speeds of m_1 and m_3.

Assume that all collisions between the spheres are elastic.



I–2. Consider a 1-dimensional quantum system with the Hamiltonian

$$\hat{H} = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2\hat{x}^2 + \lambda\hat{x}\,.$$

i) Compute the expectation value of $\hat{H}$ in the Gaussian state

$$\psi(x) = A\exp(-a(x-b)^2)\ .$$

ii) Minimize $\langle\psi|\hat{H}|\psi\rangle$ with respect to a and b.

iii) Compare your results for a and b at the minimum of $\langle\psi|\hat{H}|\psi\rangle$ with the corresponding values of the exact ground state wave function.

I–3. Consider two long parallel conducting wires extended along the $\hat{z}$ axis and separated by distance d. The radius of each wire is $r_0 \ll d$ so you may assume that the electric field at the surface of the wire has nearly azimuthal symmetry about the wire axis. Find ρ_c, the capacitance per unit length of the pair of wires.

I–4. Consider an ideal gas of atoms at chemical potential $\mu = -1eV$ and a temperature given by $k_B T = 0.1eV$. The gas is in equilibrium with a metal surface with isolated binding sites for the atoms. In each binding site there can be 0, 1, or 2 attached atoms:

- the energy with 0 attached atoms is $E = 0$
- the energy with 1 attached atom is $E = -1eV$
- the energy with 2 attached atoms is $E = -1.9eV$.

i) Determine the probability that a site has no attached atoms.

ii) Determine the average number of attached atoms at each site.

iii) If we keep the temperature of the gas the same, should we increase or decrease its pressure to have equal probabilities of 1 and 2 atoms being attached to a given site?

iv) Find the ratio by which we must increase or decrease the pressure of the gas (held at constant temperature) to have equal probabilities of 1 and 2 atoms being attached at a given site.

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II–1. A pendulum of length b and mass bob m is oscillating at small angles when the length of the pendulum string is shortened with velocity α ($\frac{db}{dt} = -\alpha$). Determine the Lagrangian equations of motion for this pendulum.

II–2. Consider a spin-1/2 system with magnetic moment $\vec{\mu} = \mu_0 \vec{\sigma}$ located in a uniform time-independent magnetic field B_0 in the positive z direction. For the time interval $0 < t < T$ an additional uniform time-independent field B_1 is applied in the positive x direction. During this interval , the system is again in a uniform constant magnetic field, but of different magnitude and direction z' from the initial one. At and before $t = 0$ the system is in the $m = 1/2$ state with respect to the z-axis.

i) At $t = 0+$, what are the probabilities for finding the system with spin projections $m' = \pm 1/2$ with respect to the z' direction ?

ii) What is the time development of the energy eigenstates with respect to the z' direction, during the time interval $0 < t < T$?

iii) At $t = T$, what is the probability amplitude for observing the system in the spin state $m = -1/2$ along the original z-axis?

Express your answers in terms of the angle θ between the z and z' axes and the frequency $\omega_0 = \mu_0 B_0/\hbar$. Note $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ denote the Pauli matrices.

II–3. A sphere of radius a is uniformly magnetized and is equivalent to a perfect magnetic dipole of magnitude M_0 pointing in the positive z direction.

i) Find the vertical and the radial components of the magnetic field.

ii) A horizontal rigid ring of mass m, radius b and resistance R is located with its center along the vertical axis at a distance $z_0 > a$ above the center of the sphere and it is released from rest. Find an expression for the current in the ring as a function of time, assuming that the ring falls for a short time with acceleration due to gravity and that it has vanishing self-inductance.

iii) Assuming again that the ring has no self-inductance but accounting for the magnetic force on the induced current, find the differential equation for the position, velocity and acceleration for short times.

II–4. A simple theory of the thermodynamics of a ferromagnet uses the free energy F written as a function of the magnetization M as

$$F = -HM + F_0 + A(T - T_c)M^2 + BM^4 \ ,$$

where H is the magnetic field, F_0, A, B, T_c are positive constants and T is the temperature.

i) What condition on the free energy F determines the thermodynamically most probable value of the magnetization M at equilibrium?

ii) Determine the value of M for $T > T_c$.

iii) For $H = 0$, find the values of M for $T > T_c$ and $T < T_c$ such that the equilibrium is stable.

I - 1

i) The condition is that m_2 bounces back:

$$\boxed{m_2 < m_3}$$

writing the conservation laws for momentum and energy for the first collision, we find that m_3 moves to the right with velocity

$$\boxed{V_3 = \frac{2m_2 V}{m_2 + m_3}}$$

and m_2 goes back with velocity

$$V' = \frac{m_3 - m_2}{m_3 + m_2} V$$

Therefore, the final velocity of m_1, moving to the left will be

$$\boxed{V_1 = \frac{2m_2}{m_1 + m_2} \frac{m_3 - m_2}{m_3 + m_2} V}$$

ii) Asymptotically, and when no collisions occur anymore, both m_1 and m_3 are moving faster than m_2. Being much lighter, this means that the momentum and energy carried by m_2 is negligible. Therefore, we may write the conservation laws as

$$\begin{cases} m_3 V_3 - m_1 V_1 = m_2 V \\ \\ m_3 V_3^2 + m_1 V_1^2 = m_2 V^2 \end{cases}$$

Now before solving this, note that the momenta actually add up to zero compared to the individual momenta of the masses m_1 and m_3 in this asymptotic limit; here is why:

$$\frac{\Delta P}{P_i} = \frac{m_3 V_3 - m_1 V_1}{m_i V_i} = \frac{m_2 V}{m_i V_i} = \frac{1}{V} \frac{m_3 V_3^2 + m_1 V_1^2}{m_i V_i} \sim \frac{V_i}{V} \to 0$$

where V_i and m_i are the typical velocities and masses of the two lateral balls. Now we can easily solve the equations as

$$\boxed{V_i = \frac{V}{m_i} \sqrt{\frac{m_1 m_2 m_3}{m_1 + m_3}} \quad \text{for } \; i = 1, 3}$$

I - 2

i) Comparing with the normal distribution we find

$$\langle \psi | \, x \, | \psi \rangle = b \quad ; \quad \langle \psi | \, (x - b)^2 \, | \psi \rangle = \frac{1}{4a}$$

On the other hand

$$\psi'' = -2a(1 - 2a(x - b)^2)\psi$$

Therefore

$$\boxed{E(a, b) = \langle \psi | \, H \, | \psi \rangle = \frac{a}{2m} + \frac{1}{2} m\omega^2 (b^2 + \frac{1}{4a}) + \lambda b}$$

ii)

$$b^* = \frac{-\lambda}{\omega^2 m} \quad ; \quad a^* = \frac{\omega m}{2}$$

with

$$E(a^*, b^*) = \frac{\omega}{2} - \frac{\lambda^2}{2\omega^2 m}$$

iii)

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2(x - b^*)^2 - \frac{1}{2}m\omega^2 b^{*2}$$

and therefore the minimum energy is

$$E_0 = \frac{\omega}{2} - \frac{\lambda^2}{2\omega^2 m}$$

I - 3

Assuming they have uniform per unit length charge density λ, we get

$$\frac{1}{2}\Delta V = \frac{\lambda}{2\pi\varepsilon_0}\log(d/r_0)$$

therefore

$$\boxed{c = \frac{\pi\varepsilon_0}{\log(d/r_0)}}$$

I - 4

i)

$$\mathcal{Z} = \sum_{n=0}^{2} e^{-\beta E_n + n\beta\mu} = e^{-10} + e^{-10} + e^{-11}$$

Therefore

$$P_0 = P_1 = \frac{e}{2e+1} \quad ; \quad P_2 = \frac{1}{2e+1}$$

ii)

$$\langle n \rangle = \sum_n nP_n = \frac{e+2}{2e+1}$$

iii, iv) For this to happen, we need

$$E_1 - \mu = E_2 - 2\mu \quad \Rightarrow \quad \mu = -0.9\,eV$$

So we need to increase μ by $0.1\,eV$. Looking at the differential $-SdT + VdP + \mu dN$, and noting that $\mu = \mu(P,T)$, we get

$$(\frac{\partial \mu}{\partial P})_T = (\frac{\partial V}{\partial N})_{P,T} = \frac{kT}{P}$$

Therefore

$$\mu = \mu_0(T) + kT\log(P)$$

Meaning that for $kT = 0.1\,eV$, we need to increase the pressure to $\boxed{P' = eP}$.

II - 1

$$L = \frac{m}{2}\left(b^2(t)\dot{\phi}^2 + \dot{b}^2(t)\right) + mgb(t)\cos\phi$$

The equation of motion is

$$\boxed{\ddot{\phi} - \frac{2\alpha}{b(t)}\dot{\phi} + \frac{g}{b(t)}\sin\phi = 0}$$

II - 2

i) It's still pointing in $+z$ direction, therefore

$$\boxed{P_{\pm} = \frac{1 \pm \cos\theta}{2}}$$

ii) It rotates around the $\hat{\mathbf{z}}'$ axis with angular velocity $\omega_0 = \mu_0\sqrt{B_0^2 + B_1^2}/\hbar$.

iii)

$$\boxed{P_{\pm} = \frac{1}{2}\left[1 \pm \left(\cos^2\theta + \sin^2\theta\,\cos\omega_0 T\right)\right]}$$

II - 3

i) In the absence of free current sources, the magnetic field $\mathbf{H} = \mu_0^{-1}\mathbf{B} - \mathbf{M}$ can be written as $\mathbf{H} = -\boldsymbol{\nabla}\psi$. With

$$\psi(r,\theta) = \sum_n A_n P_n(\cos\theta)\ \times \begin{cases} (\frac{r}{a})^n & r \le a \\ (\frac{a}{r})^{n+1} & a \le r \end{cases}$$

The boundary condition $\boldsymbol{\nabla}\cdot\mathbf{B} = 0$ gives

$$A_n = \begin{cases} \frac{1}{3}M_0 a & n = 1 \\ 0 & o.w. \end{cases}$$

Therefore

$$\boxed{\mathbf{B} = \frac{\mu_0 M_0}{3}\begin{cases} 2\hat{\mathbf{z}} & r \le a \\ \frac{a^3}{r^3}(2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\boldsymbol{\theta}}) & a \le r \end{cases}}$$

ii, iii) Let's start with the flux as a function of the position:

$$\Phi(z) = \frac{\pi}{3}\mu_0 M_0 a^3 \int_{z^2}^{z^2+b^2} \frac{3z^2 - x}{x^{2.5}}dx = \frac{2\pi\mu_0 M_0 a^3 b^2}{3(b^2+z^2)^{3/2}}$$

Therefore, the current would be

$$I = -\frac{1}{R}\frac{d\Phi}{dz}\frac{dz}{dt} = \frac{2\pi\mu_0 M_0 a^3 b^2}{R(b^2+z^2)^{5/2}}z\frac{dz}{dt}$$

For the case of a free fall:

$$\boxed{I(t) = \frac{-2\pi\mu_0 M_0 a^3 b^2}{R[b^2 + (z_0 - \frac{1}{2}gt^2)^2]^{5/2}}(z_0 - \frac{1}{2}gt^2)gt}$$

Of course, there is a resistive force due to the current, with a z component

$$F = -2\pi b I B_s = -\frac{4\pi^2 b^4 \mu_0^2 M_0^2 a^6}{R(b^2+z^2)^5} z^2 \frac{dz}{dt}$$

the differential equation is then

$$\boxed{m\frac{d^2z}{dt^2} + \frac{(2\pi\mu_0 M_0 b^2 a^3)^2}{R} \frac{z^2}{(b^2+z^2)^5} \frac{dz}{dt} = 0}$$

II - 4

i) The most probable M corresponds to the minimum value of the free energy F as a function of (T, M).

ii) The unique equilibrium value of M is found by solving

$$\boxed{M^3 + \frac{A(T-T_c)}{2B}M - \frac{H}{4B} = 0}$$

iii) For $H = 0$:

$$\frac{\partial F}{\partial M} = 2A(T-T_c)M + 4BM^3$$

when $T > T_c$, the only solution is $M = 0$ and it is stable since $\frac{\partial^2 F}{\partial M^2}|_{M=0} = 2A(T-T_c) > 0$. When $T < T_c$, the solution at $M = 0$ still exists but is unstable. Instead, the two solutions at

$$M_{\pm} = \pm\sqrt{\frac{A(T_c - T)}{2B}}$$

These are stable since

$$\frac{\partial^2 F}{\partial M^2}|_{M_\pm} = 2A(T_c - T) > 0$$