PSU Physics PhD Qualifying Exam Solutions
Spring 2021

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September 23, 2022

Qualifying Exam for Ph.D. Candidacy
Department of Physics
February 6, 2021

Part I

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Boltzmann's constant	k_B	$1.381 \times 10^{-23}\,\text{J}\,\text{K}^{-1}$
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Gas constant	R	$8.314\,\text{J}\,\text{mol}^{-1}\,\text{K}^{-1}$
Planck's constant	h $\hbar = h/2\pi$	$6.626 \times 10^{-34}\,\text{J}\,\text{s}$ $1.055 \times 10^{-34}\,\text{J}\,\text{s}$
Speed of light in vacuum	c	$2.998 \times 10^{8}\,\text{m}\,\text{s}^{-1}$
Permittivity constant	ϵ_0	$8.854 \times 10^{-12}\,\text{F}\,\text{m}^{-1}$
Permeability constant	μ_0	$1.257 \times 10^{-6}\,\text{N}\,\text{A}^{-2}$
Gravitational constant	G	$6.674 \times 10^{-11}\,\text{m}^3\,\text{kg}^{-1}\,\text{s}^{-2}$
Standard atmospheric pressure	1 atmosphere	$1.01 \times 10^{5}\,\text{N}\,\text{m}^{-2}$
Stefan-Boltzmann constant	σ	$5.67 \times 10^{-8}\,\text{W}\,\text{m}^{-2}\,\text{K}^{-4}$
Electron rest mass	m_e	$9.109 \times 10^{-31}\,\text{kg} = 0.5110\,\text{MeV}\,c^{-2}$
Proton rest mass	m_p	$1.673 \times 10^{-27}\,\text{kg} = 938.3\,\text{MeV}\,c^{-2}$
Origin of temperature scales		$0\,°\text{C} = 273\,\text{K}$
1 large calorie (as in nutrition)		$4.184\,\text{kJ}$
1 GeV		$1.609 \times 10^{-10}\,\text{J}$

Definite integrals:

$$\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}. \tag{I–1}$$

$$\int_0^\infty x^n e^{-x} dx = \Gamma(n+1) = n!. \tag{I–2}$$

Indefinite integrals:

$$\int \frac{x}{(x^2+a^2)^n} dx = \frac{1}{2(1-n)} \frac{1}{(x^2+a^2)^{n-1}} n + c \quad \text{for } n \neq 0, 1 \tag{I–3}$$

Gradient in spherical polar coordinates (r, θ, ϕ):

$$\vec{\nabla} = \vec{e}_r \frac{\partial}{\partial r} + \vec{e}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \vec{e}_\phi \frac{1}{r \sin\theta} \frac{\partial}{\partial \phi}\,, \tag{I–4}$$

Laplacian in spherical polar coordinates (r, θ, ϕ):

$$\nabla^2 f = \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 f}{\partial \phi^2}. \tag{I–5}$$

Laplacian in cylindrical coordinates (r, θ, z):

$$\nabla^2 f = \frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial f}{\partial r}\right) + \frac{1}{r^2} \frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}. \tag{I–6}$$

Pauli matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \; . \tag{I–7}$$

I–1. Two Lagrangians L' and L, which differ by the total time derivative dF/dt of some function $F(q,t)$,

$$L' = L + dF/dt \; ,$$

are physically equivalent, i.e. they lead to the same Lagrange's equations of motion.

(a) What is the relation between the generalized momenta p' and p which these two Lagrangians yield?

(b) What is the relation between the Hamiltonians H' and H which these two Lagrangians yield?

(c) Show explicitly that Hamiltonian's equations of motion in the primed quantities are equivalent to those in the unprimed quantities.

I–2. In this problem, we consider the optical transition rate of light through graphene. Electrons in graphene are described by two-dimensional massless Dirac fermions with the Hamiltonian $H_0 = \hbar\nu_f(k_x\sigma_x + k_y\sigma_y)$, where σ_x and σ_y are 2×2 Pauli matrices and k_x and k_y are the components of the wave vector.

a) Find the energies $E_\pm(\vec{k})$, of the Hamiltonian H_0, as well as the corresponding eigenstate wave functions, denoted as $\psi_\pm$.

b) Consider the coupling of Dirac electrons with electromagnetic fields with the full Hamiltonian

$$H = \hbar\nu_f((k_x + \frac{e}{\hbar}A_x)\sigma_x + (k_y + \frac{e}{\hbar}A_y)\sigma_y) \; .$$

Denote $H_1 = e\nu_f(A_x\sigma_x + A_y\sigma_y)$, where $\vec{A} = (A_x, A_y)$ is the vector potential of the light, and calculate the transition matrix element

$$M_{+-}(\vec{k}) = \langle\psi_+|H_1|\psi_-\rangle \; .$$

Assume that the vector potential $\vec{A}$ is that to light with frequency ω, and the corresponding electric field given by $\vec{E}(t) = \vec{E}_0 e^{i\omega t}$.

c) Now let's assume that $E_-(\vec{k}) < E_+(\vec{k})$ and that the $E_-(\vec{k})$ states are always occupied while the $E_+(\vec{k})$ states are unoccupied. Use the Fermi's golden rule to calculate the optical transition rate Γ of Dirac electrons in graphene. Recall that the Fermi's golden rule is given by

$$\Gamma = \frac{2\pi}{\hbar}\int \frac{d^2k}{(2\pi)^2}|M_{+-}(\vec{k})|^2\delta(E_+ - E_- - \hbar\omega) \; ,$$

where ω is the frequency of the light.

I–3. Two bodies have internal energy given by $U = NCT$. The number of particles N and the specific heat C are the same for each body; the initial temperatures of the two bodies are T_1 and T_2. The two bodies are used to drive a Carnot engine which will bring them to a common final temperature T_f while doing work. The Carnot cycle, comprised of quasistatic isotermal transformations and two quasistatic adiabatic transformations, is sketched in figure I-3.1.

a) Find the final temperature T_f.

b) Find the work done by the engine.

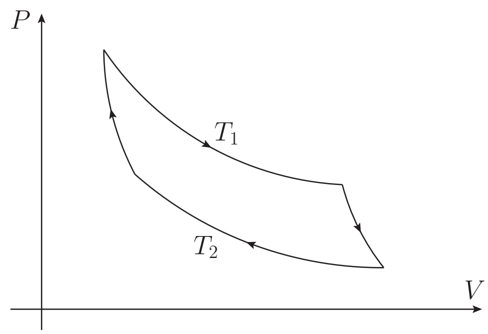

Figure I-3.1 The Carnot cycle.

I–4. Consider a parallel plate capacitor in vacuum, with circular plates each of radius a, separated by a distance d, see Figure I-4.1.

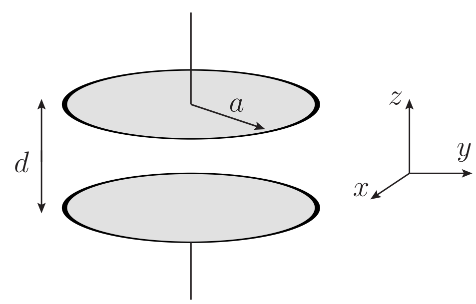

Figure I-4.1 Circular capacitor.

A current I is slowly charging the capacitor. Using the Poynting vector, defined in terms of the electric and magnetic fields $\vec{E}$ and $\vec{B}$ by

$$\vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B}$$

and representing the directional energy flux of an electromagnetic field, find the rate at which the electromagnetic field feeds energy into the capacitor. Show that the energy input is also equal to IV where V is the potential difference between the plates. Assume that the electric field is uniform out to the edges of the plates.

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II–1. A relativistic proton collider consists of two storage rings, each with 100 GeV protons orbiting in the horizontal plane in the presence of nearly constant vertical magnetic fields. There is a slight displaced between these nearly circular orbits with one beam orbiting clockwise and the other counter-clockwise. Each ring has of radius $R_{\text{ring}} = 1\,km$ and contains 110 proton pulses (called bunches). At time $t = 0$, each bunch contains 10^{11} protons. At the time they collide the protons are randomly located within a cylindrical shape of length $L = 1\,m$ and radius $R = 0.1\,mm$. The cylinder axis is always concentric with the local beam axis. Bunch collisions only occur within a $1\,m$ long interaction region along the ring circumference, where the circular beam orbits have been slightly perturbed so the bunches in this region move along a single common collision axis. A given bunch in one beam always collides with a particular bunch in the other beam. All proton collisions occur within the collision region.

At the interaction point, all protons in a beam bunch have equal momentum vectors parallel to the beam line and the opposite that of the other colliding beam bunch.

a) Absorption Cross section: Assume a proton passing through a bunch breaks up if its trajectory takes it to within a transverse distance of $d = 10^{-15}\,m$ from a proton in the opposing bunch. What is the rate of interaction within a single bunch?

b) Assume the decay of the number of protons in a bunch is only due to these absorptive collisions. Derive the expression for the number of protons in a bunch as a function of time, $N(t)$ in terms of variables $(R_{\text{ring}}, R, L, d)$.

(Note: $N(0) = 10^{11}$ and that each bunch in one ring only collides with a unique partner bunch in the other ring, removing 1 proton from each of the partner bunches. Assume that the speed of protons is very near the speed of light.)

II–2. A particle is initially in its ground state in a box with infinite walls at $x = 0$ and $x = L$.

a) Find the probability density for finding the particle at position $x = L/2$.

a) The wall at $x = L$ is suddenly moved to $x = 2L$. Find the probability that the particle is found in the second excited state of the expanded box.

b) Suppose that the walls of the original box, $x \in [0, L]$, are suddenly dissolved while the particle was in its ground state. Find the probability that the freed particle has momentum p.

II–3. Consider a rigid lattice of indistinguishable spin-1/2 atoms in a magnetic field H. Each spin has two states, with energies $+\mu_0 H$ and $-\mu_0 H$ for spins up and down, respectively. The system is at temperature T.

a) Find the canonical partition function.

b) Determine the total induced magnetic moment of the system.

c) Determine the entropy of the system.

II–4. A controlled voltage source, shown in Figure II-4.1, produces a voltage signal at output A. The source can be in one two states, shown in Figure II-4.2.

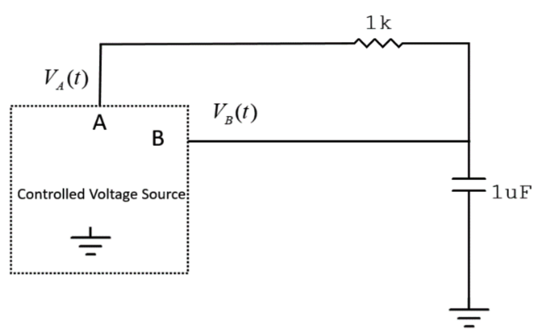

Figure II-4.1 Controlled voltage source.

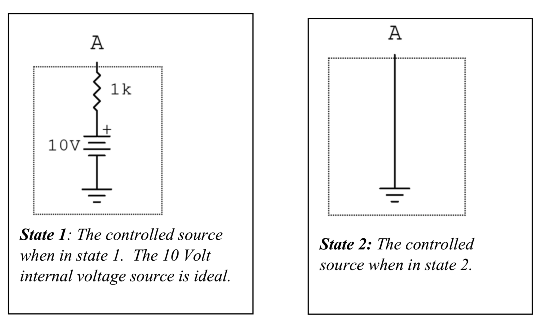

Figure II-4.2 The two possible states of the controlled voltage source.

When the voltage source is in state 1, the source provides 10 volts between output A and ground, with 1000 ohms of output impedance as shown. When the voltage source is in state 2, it provides a direct short to ground. The input B, which has infinite input impedance and is used to trigger a change of state.

• When the voltage at B increases and passes through 7 volts, the state is set to "state 2".

• When the voltage at B is decreases and passes through 2 volts, the state is set to "state 1".

a) Determine the differential equation for the charge on the capacitor, $Q(t)$, when the source is in state 1.

b) Determine the differential equation for $V_B(t)$ when the source is in state 2. Write down the general solution for $V_B(t)$.

c) Determine the time for a full cycle state 1 to state 1.

I-1

a)

$$p_i' = \frac{\partial L'}{\partial \dot{q}_i} = \frac{\partial L}{\partial \dot{q}_i} + \frac{\partial}{\partial \dot{q}_i}(\frac{\partial F}{\partial t} + \sum_j \frac{\partial F}{\partial q_j}\dot{q}_j) = p_i + \frac{\partial F}{\partial q_i}$$

b)

$$H' = p_i'\dot{q}_i - L' = (p_i + \frac{\partial F}{\partial q_i})\dot{q}_i - L - \frac{\partial F}{\partial t} - \frac{\partial F}{\partial q_i}\dot{q}_i = H - \frac{\partial F}{\partial t}$$

c) Let's start with the EOM in the primed formulation

$$\frac{dq_i}{dt} = \frac{\partial H'}{\partial p_i'}$$

To find the partial derivative, note that keeping (q_i, t) constant, we have

$$dp_i' = dp_i \quad ; \quad dH' = dH$$

Therefore

$$\frac{\partial H'}{\partial p_i'} = \frac{\partial H}{\partial p_i}$$

and therefore we prove the equivalence for the first set of equations. Next

$$\frac{dp_i'}{dt} = \frac{dp_i}{dt} + \frac{\partial^2 F}{\partial q_i \partial t} + \frac{\partial^2 F}{\partial q_i \partial q_j}\dot{q}_j = -\frac{\partial H'}{\partial q_i}$$

This time, to evaluate the derivative on the right hand side, note that $dp_j' = dt = 0$ and therefore

$$\delta p_j = -\frac{\partial^2 F}{\partial q_j \partial q_i}\delta q_i$$

and

$$\delta H' = \frac{\partial H}{\partial q_i}\delta q_i - \frac{\partial H}{\partial p_j}\frac{\partial^2 F}{\partial q_i \partial q_j}\delta q_i - \frac{\partial^2 F}{\partial t \partial q_i}\delta q_i$$

Finally this leads to

$$\frac{dp_i'}{dt} = \frac{dp_i}{dt} + \frac{\partial^2 F}{\partial q_i \partial t} + \frac{\partial^2 F}{\partial q_i \partial q_j}\dot{q}_j = -\frac{\partial H}{\partial q_i} + \frac{\partial H}{\partial p_j}\frac{\partial^2 F}{\partial q_i \partial q_j} + \frac{\partial^2 F}{\partial t \partial q_i}$$

And this is clearly equivalent to

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}$$

I-2

I change ν_f to ω_f.

a)

$$H = \omega_f \begin{pmatrix} 0 & k_x - ik_y \\ k_x + ik_y & 0 \end{pmatrix}$$

Which is diagonalized as

$$\psi_\pm = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \pm e^{i\alpha} \end{pmatrix} \exp\left[i(k_x x + k_y y)\right]/\sqrt{2\pi}$$

where

$$\alpha \equiv \arg(k_x + ik_y)$$

And the energies are

$$E_\pm = \pm\omega_f\sqrt{k_x^2 + k_y^2}$$

b) Neglecting the spatial dependence of the vector potential, this is given by

$$M = \frac{\omega_f e}{2}(1 \quad e^{-i\alpha})\begin{pmatrix} 0 & A_x - iA_y \\ A_x + iA_y & 0 \end{pmatrix}\begin{pmatrix} 1 \\ -e^{i\alpha} \end{pmatrix}$$

$$= ie\omega_f \operatorname{Im}(Ae^{-i\alpha}) = ie\omega_f|\mathbf{A}|\sin(\varphi)$$

Where φ is the angle between $\mathbf{k}$ and $\mathbf{A}$ vectors.

c) First of all, let's note that

$$|\mathbf{E}| = c|\mathbf{B}| = kc|\mathbf{A}| = \omega|\mathbf{A}|$$

And therefore

$$M = ie\frac{\omega_f}{\omega}E_0\sin\varphi$$

Now we can do the integration as

$$\Gamma = 2\pi\int\frac{d^2\mathbf{k}}{(2\pi)^2}\frac{e^2\omega_f^2E_0^2}{\omega^2}\sin^2\varphi\,\delta\Big(2\omega_f|\mathbf{k}| - \omega\Big)$$

$$= \frac{e^2E_0^2\omega_f}{4\pi\omega^2}\int kdk\,d\varphi\,\sin^2\varphi\,\delta\Big(k - \frac{\omega}{2\omega_f}\Big) = \boxed{\frac{e^2E_0^2}{8\omega}}$$

I-3

a, b) If the engine takes a small amount of heat $đQ$ from T_1, then

$$dT_1 = -\frac{đQ}{NC}$$

$$đW = (1 - T_2/T_1)đQ$$

$$dT_2 = \frac{T_2 đQ}{NCT_1}$$

From the first and the third equation

$$dT_2 = -\frac{T_2}{T_1}dT_1$$

Therefore T_1T_2 is a constant and the final temperature is given by

$$\boxed{T_f = \sqrt{T_1T_2}}$$

Then the work is given by

$$W = -NC\int_{T_1}^{\sqrt{T_1T_2}}dx\left(1 - \frac{T_1T_2}{x^2}\right)$$

$$= \boxed{NC\Big(T_1 + T_2 - 2\sqrt{T_1T_2}\Big)}$$

I-4

Neglecting edge effects and in the adiabatic regime, the electric field is given by

$$\mathbf{E} = \frac{V(t)}{d}\hat{\mathbf{z}}$$

everywhere. Then the magnetic field is found by the Ampere's law as

$$\mathbf{B} = \frac{\mu_0\varepsilon_0}{2d}\frac{dV}{dt}s\hat{\boldsymbol{\varphi}}$$

Therefore the Poynting vector is

$$\mathbf{S} = -\frac{\varepsilon_0 V\dot{V}s}{2d^2}\hat{\mathbf{s}}$$

The energy is moving inside the capacitor volume at a rate

$$\dot{U}_{\text{in}} = 2\pi ad \times |\mathbf{S}| = \frac{\pi\varepsilon_0 a^2}{2d}\frac{d}{dt}V^2 = \frac{d}{dt}\left(\frac{1}{2}\frac{\pi a^2\varepsilon_0}{d}V^2\right) = \frac{d}{dt}(\frac{1}{2}CV^2) = V(C\dot{V}) = V\dot{Q} = VI$$

This proves everything that we wanted to prove.

II-1

a) The probability that a proton passes an opposing bunch without breaking down is

$$\mathbb{P}[\text{no interation}] = \exp\left(-n\frac{\sigma L}{AL}\right) = e^{-N\sigma/A} \approx 1 - N \times (\frac{d}{2R})^2 = 1 - 5\times 10^{-12}$$

b) The bunch-bunch collisions occur at a rate

$$f = \frac{c}{2\pi R_{\text{ring}}}$$

Therefore, after some time ΔT passes, $f\Delta T$ bunch-bunch collisions occur and

$$\Delta N = -N \times \frac{c\Delta T}{2\pi R_{\text{ring}}} \times N \times (\frac{d}{2R})^2$$

This leads to the differential equation

$$\frac{dN}{dt} = -(\frac{cd^2}{8\pi R^2 R_{\text{ring}}})N^2$$

solved as

$$\boxed{N(t) = \frac{N_0}{1 + \frac{N_0 cd^2}{8\pi R^2 R_{\text{ring}}}t} \approx \frac{10^{11}}{1 + 1.2\times 10^{-7}s^{-1}\times t}}$$

II-2

a) The wave function is

$$\psi_1 = \sqrt{\frac{2}{L}}\sin\left(\frac{\pi x}{L}\right)$$

and therefore

$$\rho(L/2) = |\psi_1(L/2)|^2 = \boxed{\frac{2}{L}}$$

b) The wave function does not follow the change immediately; thus we can find the overlap as

$$\langle 2|\psi_1\rangle = \frac{\sqrt{2}}{L}\int_0^L \sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{\pi x}{L}\right)dx = \frac{1}{\sqrt{2}}$$

and the probability is

$$\boxed{P_2 = \frac{1}{2}}$$

c) Once again, the wave function does not follow the quick change and the probability density amplitude is given by (not the probability)

$$\begin{aligned}\phi(p) &= \frac{1}{\sqrt{\pi L}}\int_0^L \sin\Big(\frac{\pi x}{L}\Big)e^{-ipx}dx \\ &= 2\sqrt{\pi L}e^{-ipL/2}\frac{\cos(pL/2)}{\pi^2 - p^2L^2}\end{aligned}$$

The probability density is then

$$\boxed{\rho(p) = \frac{4\pi L}{(\pi^2 - p^2L^2)^2}\cos^2(pL/2)}$$

II-3

a)

$$Z = \sum_{\mathbf{x}\in\{\pm 1\}^n}\exp\left(\beta\mu_0 H\sum_i x_i\right) = \Big[2\cosh(\beta\mu_0 H)\Big]^N$$

b)

$$\langle M\rangle = \gamma\frac{\hbar}{2}\langle\sum_i x_i\rangle = \boxed{\gamma\frac{\hbar}{2}N\,\tanh(\beta\mu_0 H)}$$

c)

$$S = -\beta E + \log Z = \boxed{N\Big\{-\beta\mu_0 H\tanh(\beta\mu_0 H) + \log\big[2\cosh(\beta\mu_0 H)\big]\Big\}}$$

II-4

Since I prefer working with symbols, let's define

$$R \equiv 1\,k\Omega \quad ; \quad C \equiv 1\,\mu F \quad ; \quad V_0 = 10\,V \quad ; \quad V_- = 2\,V \quad ; \quad V_+ = 7\,V$$

a)

$$\boxed{\frac{dQ}{dt} = \frac{1}{2R}(V_0 - \frac{Q}{C})}$$

b)

$$\boxed{C\frac{dV_B}{dt} = -\frac{V_B}{R}}$$

c) When V_B hits $V_- = 2\,V$, the slope will be positive as

$$\dot{V}_B = \frac{1}{2RC}(V_0 - V_B)$$

suggests. The solution is

$$V_B(t) = V_0 - (V_0 - V_-)e^{-t/2RC}$$

Then, when the voltage hits V_+ at time

$$t_1 = 2RC\log\left(\frac{V_0 - V_-}{V_0 - V_+}\right)$$

it gets into state 2 and starts decreasing according to

$$\dot{V}_B = -V_B/RC$$

solved as

$$V_B(t) = V_+ e^{-(t-t_1)/RC}$$

Now we can see when it completes its circle of life and comes down to V_- again:

$$T = t_1 + RC\log\left(\frac{V_+}{V_-}\right) = \boxed{RC\log\left[\frac{V_+(V_0 - V_-)^2}{V_-(V_0 - V_+)^2}\right] \approx 3.2\,ms}$$