A Solution Manual to Eric Poisson's A Relativist's Toolkit

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My units are such that $c = 4\pi G = 1$; this may lead to some discrepancies with the book.

Chapter 1

1

a)

$$ds^2 = dr^2 + r^2 \sin^2 \alpha d\varphi^2$$

b)

$$x = r\cos(\varphi \sin\alpha) \quad ; \quad y = r\sin(\varphi \sin\alpha)$$

No, as r varies from 0 to ∞, and φ ranges in $[0, 2\pi]$, only parts of the 2D $x - y$ plane with $\arg(x, y) \leq 2\pi \sin\alpha$.

c) This is best seen in the $x-y$ coordinate system, the vector is parallel transported along the portion of the circle with $0 \leq \theta \leq 2\pi \sin\alpha$; then it teleports across the wedge back to where it started its journey. During this last maneuver, the vector picks an extra angle, equal to the wedge opening, $2\pi(1-\sin\alpha)$ in the same direction of the journey.

2

$$\frac{Du^\alpha}{d\lambda^*} = \frac{dx^\beta}{d\lambda^*}\nabla_\beta \frac{dx^\alpha}{d\lambda^*} = \frac{d\lambda}{d\lambda^*} t^\beta \nabla_\beta \frac{d\lambda}{d\lambda^*} t^\alpha = (\frac{d\lambda}{d\lambda^*})^2 t^\beta \nabla_\beta t^\alpha + \frac{d\lambda}{d\lambda^*} t^\alpha t^\beta \nabla_\beta \frac{d\lambda}{d\lambda^*}$$
$$= t^\alpha \frac{d\lambda}{d\lambda^*} \left(\kappa + \frac{d}{d\lambda}\right) (\frac{d\lambda^*}{d\lambda})^{-1} = 0 \quad \blacksquare$$

3

a)

$$\frac{d}{d\lambda}\varepsilon = t^\mu \nabla_\mu (-t^\alpha t_\alpha) = -2t_\alpha t^\mu \nabla_\mu t^\alpha = -2\kappa t^\alpha t_\alpha = 2\kappa\varepsilon$$

leading to

$$\varepsilon(\lambda) = \varepsilon(\lambda_0) \exp\left(2\int_{\lambda_0}^{\lambda} d\lambda' \, \kappa(\lambda')\right)$$

b)

$$\frac{dp}{d\lambda} = t^\mu \nabla_\mu (\xi_\alpha t^\alpha) = \frac{1}{2} t^\mu t^\alpha \nabla_{\{\mu} \xi_{\alpha\}} + \kappa \xi_\alpha t^\alpha = \kappa p$$

solved as

$$p(\lambda) = p(\lambda_0) \, \exp\left(\int_{\lambda_0}^{\lambda} d\lambda' \, \kappa(\lambda') \right)$$

c)

$$\frac{dq}{d\tau} = u^\mu \nabla_\mu (b_\alpha u^\alpha) = \frac{1}{2} u^\mu u^\alpha \nabla_{\{\mu} b_{\alpha\}} = -c$$

therefore

$$q(\tau) = q(\tau_0) - \int_{\tau_0}^{\tau} d\tau' \, c\,(x^\alpha(\tau'))$$

4

Let's use the Leibniz property only, assuming $\pounds_u T_{\mu\nu} = S_{\mu\nu}$, we have

$$u^\alpha \nabla_\alpha (T_{\mu\nu} A^\mu B^\nu) = \pounds_u (T_{\mu\nu} A^\mu B^\nu) = S_{\mu\nu} A^\mu B^\nu + T_{\mu\nu} B^\nu (u^\alpha \nabla_\alpha A^\mu - A^\alpha \nabla_\alpha u^\mu) + T_{\mu\nu} A^\mu (u^\alpha \nabla_\alpha B^\nu - B^\alpha \nabla_\alpha u^\nu)$$

$$\Rightarrow \quad A^\mu B^\nu \left(u^\alpha \nabla_\alpha T_{\mu\nu} - S_{\mu\nu} + T_{\alpha\nu} \nabla_\mu u^\alpha + T_{\mu\alpha} \nabla_\nu u^\alpha \right) = 0 \;\; \forall A, B$$

Which leads to

$$\pounds_u T_{\mu\nu} = u^\alpha \nabla_\alpha T_{\mu\nu} + T_{\alpha\nu} \nabla_\mu u^\alpha + T_{\mu\alpha} \nabla_\nu u^\alpha$$

Chapter 2

Chapter 3

1

Let's work in the units with $r_S = 2GM/c^2 = 1$.

a) The gradient is

$$dx^\mu \partial_\mu T = dt + \frac{\sqrt{r}\, dr}{r-1}$$

Interestingly, this is normal and timelike

$$\boxed{n_\alpha = \left(1, \frac{\sqrt{r}}{r-1}, 0, 0\right)}$$

The parametric equations are

$$\boxed{t = T - 2\Big[\sqrt{R} + \frac{1}{2} \log\Big(\frac{\sqrt{R}-1}{\sqrt{R}+1}\Big)\Big] \quad ; \quad r = R \quad ; \quad \theta = \Theta \quad ; \quad \phi = \Phi}$$

where (R, Θ, Φ) are the tangent coordinates.

b) The induced metric is flat

$$\boxed{ds^2 = dR^2 + R^2 d\Omega^2}$$

c) Let's start with the covariant derivative $\nabla_\mu n_\nu$:

$$\nabla_t n_t = \frac{-1}{2r^{5/2}}$$

$$\nabla_t n_r = \nabla_r n_t = \frac{-1}{2r(r-1)}$$

$$\nabla_r n_r = \frac{-\sqrt{r}}{2(r-1)^2}$$

$$\nabla_\theta n_\theta = \sqrt{r}$$

$$\nabla_\varphi n_\varphi = \sqrt{r}\sin^2\theta$$

Then the nonzero components of the extrinsic curvature follow

$$\boxed{K_{RR} = \frac{-1}{2R^{3/2}} \quad ; \quad K_{\Theta\Theta} = \sqrt{R} \quad ; \quad K_{\Phi\Phi} = \sqrt{R}\sin^2\Theta}$$

This is clearly in accordance (in fact, it's the same calculation) with the results described in section 3.6.6. The trace is

$$K = h^{ab}K_{ab} = \frac{3}{2R^{3/2}}$$

Since the metric is T independent, and $n_\mu = \partial_\mu T$, the normal vector is a Killing vector. Since it has constant length, it is also tangent to a geodesic bundle; the divergence of which is given by

$$\theta = \nabla_\alpha n^\alpha = K$$

This agrees with the result in section 2.3.7 of the book as well.

d) Let's use the results from part (a) directly

$$ds^2 = -(1-1/r)dt^2 + dr^2/(1-1/r) + r^2 d\Omega^2 = -\frac{R-1}{R}\left(dT - \frac{\sqrt{R}dR}{R-1}\right)^2 + \frac{RdR^2}{R-1} + R^2 d\Omega^2$$

$$= \boxed{-dT^2 + \left(dR + dT/\sqrt{R}\right)^2 + R^2 d\Omega^2}$$

2

a) The normal is best found given the constraint description of the hypersurface. It is

$$a^2 = \eta_{AB} z^A z^B = \text{const.}$$

The normal is then found as

$$\boxed{n_A = \frac{1}{a}\eta_{AB} z^B}$$

b)

$$ds^2 = \eta_{AB} dz^A dz^B = -\cosh(t/a)dt^2 + \sum_{A>0}(dz^A)^2$$

$$= -\cosh^2(t/a)dt^2 + \sinh^2(t/a)dt^2 + a^2\cosh^2(t/a)d\Omega_3^2$$

$$= \boxed{-dt^2 + a^2\cosh^2(t/a)d\Omega_3^2}$$

This is of course, the de Sitter space time. It's conformally flat and is a solution to the Einstein field equations in vacuum with positive cosmological constant.

c)

$$K_{\alpha\beta} = e^A_\alpha e^B_\beta \nabla_A n_B = e^A_\alpha e^B_\beta \partial_A n_B$$

$$= e^A_\alpha e^B_\beta (\frac{1}{a}\eta_{AB} - \frac{1}{a^2} z_B \partial_A a) = \frac{1}{a} e^A_\alpha e^B_\beta \eta_{AB} = \boxed{\frac{1}{a} g_{\alpha\beta}}$$

The other terms vanish because on the hypersurface, a is constant. Now let's use the fully tangential component of the Gauss-Codazzi relations; (equation 3.39). It reads

$$0 = R_{\alpha\beta\mu\nu} + K_{\alpha\nu}K_{\beta\mu} - K_{\alpha\mu}K_{\beta\nu}$$

or

$$\boxed{R_{\alpha\beta\mu\nu} = \frac{1}{a^2}(g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\nu}g_{\beta\mu})}$$

3

a) The mass function is defined in the metric form

$$ds^2 = \left[1 - m(r)/2\pi r\right]^{-1} dr^2 + r^2 d\Omega^2$$

Comparison leads to

$$\boxed{m = 2\pi r\left[1 - (\frac{dr}{dl})^2\right]}$$

b) The constraint equation reads

$$^3R = 4T(n,n)$$

Or, in terms of the mass function

$$\frac{dm}{dr} = 4\pi r^2 \rho$$

The regularity at the origin imposes

$$m(0) = 0$$

Therefore

$$\boxed{m(r) = \frac{4}{3}\pi r^3 \rho}$$

Putting this back to the differential equation connecting $r$and l, we find

$$\boxed{r(l) = \sqrt{\frac{3}{2\rho}}\sin\left(\sqrt{\frac{2\rho}{3}}l\right)}$$

c, d) This is clear from the expression for $r(l)$ that it can not go beyond

$$\boxed{r_{\text{max}} = \sqrt{\frac{3}{2\rho}}}$$

Then, since $dm/dr > 0$, the maximum mass is also achieved at maximum aerial radius, when the mass function attains the value

$$\boxed{m(r) = 2\pi r_{\text{max.}}}$$

e) The metric is

$$ds^2 = dl^2 + r_{\text{max}}^2 \sin^2(l/r_{\text{max}})d\Omega_2^2$$

This space-time is symmetric under the discrete transformation

$$l \to \pi r_{\text{max}} - l$$

Therefore, the $l = \pi r_{\text{max}}$ is also a center of the polar coordinates where the area of the sphere vanishes and all the Ω_2 variables become irrelevant. This is exactly the descriptuion of a 3 sphere, $\mathbb{S}^3$. One just needs to define $\psi \equiv l/r_{\text{max}}$ to find

$$ds^2 = r_{\text{max}}^2 d\Omega_3^2$$

4

The condition $[K_{ab}] = 0$ is clearly necessary for regularity of the Riemann tensor because of how the Gauss-Codazzi equations relate some components of the Riemann tensor to the extrinsic curvature. It remains to show that $R(e_a, n, e_b, n)$ is also consistent if the extrinsic curvature is the same from both sides. Let y^a be the local normal coordinate system on the hypersurface and l be the orthogonal geodesic direction. Then, the only non-vanishing metric derivative is

$$\partial_l g_{ab} = 2K_{ab}$$

The Riemann component that we are after, then simplifies into

$$R^l{}_{alb} = \partial_l \Gamma^l_{ab} - \Gamma^l_{bc}\Gamma^c_{la}$$

$$= -\varepsilon(\partial_l K_{ab} - K_{ac}K^c_b)$$

Clearly, this shows that if $[K_{ab}] = 0$ is satisfied, the Riemann tensor will at most have a jump discontinuity and not a delta function singulairty.

5

Now that we have all of the components of the Riemann tensor, we may as well find the stress energy tensor completely by following the standard procedure.

$$T_{\alpha\beta} = \frac{1}{2} R^{\mu}{}_{\alpha\mu\beta} - \frac{1}{4} R^{\mu\nu}{}_{\mu\nu} g_{\alpha\beta}$$

The answer will be

$$T_{ll} = \frac{1}{4}\big(-\varepsilon\, {}^{3}R + K^2 - K_{ab}K^{ab}\big)$$

$$T_{la} = \frac{1}{2}(D^b K_{ab} - D_a K)$$

$$T_{ab} = {}^{3}T_{ab} + \frac{\varepsilon}{4}\big[2(2K_{ac}K^c_b - \partial_l K_{ab} - KK_{ab}) - h_{ab}(3K_{ab}K^{ab} - 2\partial_l K - K^2)\big]$$

Now we can explicitly write

$$-\varepsilon[j^a] = -\varepsilon h^{ab}[T_{lb}] = \frac{1}{2}(D_b[K^{ab}] - D^a[K]) = D_b S^{ab} \ \blacksquare$$

Let's consider a timelike shell like $z = 0$. The t-component formula above asserts that the discontinuity in T^{tz}, or the mass flow across the shell is equal to the rate with which mass accumulates on the shell. The other components of the formula are interpreted similarly.

6

I will work in the units where $l_0 = 1$. Also, the tangent coordinates are (t, θ, φ). Topologically speaking, this is the same as a stationary space with $\mathbb{S}^3$ topology. The space has two flattened hemispheres connected together via the hypersurface.

a) Let's start with finding the extrinsic curvature on both sides. The normal vector is

$$n = \partial_l$$

The nonzero Christoffel symbols are

$${}^{\pm}\Gamma^l_{\theta\theta} = \pm r \quad ; \quad {}^{\pm}\Gamma^l_{\varphi\varphi} = \pm r \sin^2\theta$$

$${}^{\pm}\Gamma^{\theta}_{l\theta} = {}^{\pm}\Gamma^{\theta}_{\theta l} = {}^{\pm}\Gamma^{\varphi}_{l\varphi} = {}^{\pm}\Gamma^{\varphi}_{\varphi l} = \frac{\mp 1}{r}$$

$$\Gamma^{\theta}_{\varphi\varphi} = -\sin\theta\,\cos\theta \quad ; \quad \Gamma^{\varphi}_{\theta\varphi} = \Gamma^{\varphi}_{\varphi\theta} = \cot\theta$$

From these, it follows that K_{ab} is only nonzero for angular components.

$${}^{\pm}K_{\theta\theta} = \mp 1 \quad ; \quad {}^{\pm}K_{\varphi\varphi} = \mp \sin^2\theta$$

Then follows S_{ab}:

$$\boxed{S_{tt} = 2 \quad ; \quad S_{\theta\theta} = -1 \quad ; \quad S_{\varphi\varphi} = -\sin^2\theta}$$

This corresponds to a surface density σ, surface pressure p, and 4 velocity V as below

$$\boxed{V = \partial_t \quad ; \quad \sigma = 2 \quad ; \quad p = -1}$$

b) The null tangent vector is $k = \partial_t + \partial_l$. The expansion is

$$^{\pm}\theta = \nabla_\alpha k^\alpha = \partial_\alpha k^\alpha + {}^{\pm}\Gamma^\alpha_{\mu\alpha} k^\mu = {}^{\pm}\Gamma^\alpha_{l\alpha} = \boxed{\frac{\mp 2}{r}}$$

This clearly changes sign from positive to negative as the geodesic crosses from the negative region to the positive region.

c) Raychaudhuri's equation is

$$\frac{d\theta}{d\lambda} = -B_{\alpha\beta}B^{\beta\alpha} - R_{\mu\nu}k^\mu k^\nu$$

Integrating this across the shell, it follows that

$$^{+}\theta - {}^{-}\theta = -\int_{1-\varepsilon}^{1+\varepsilon} dl\, R(\partial_t + \partial_l, \partial_t + \partial_l) = -2S_{ab}k^a k^b = -2S_{tt} = -4$$

Which is in accordance with the explicit result we found.

7

a) The first junction condition, implies that the hypersurface is described by functions

$$r^- = r^+ = R(\tau) \quad ; \quad t^- = t^-(\tau) \quad ; \quad t^+ = t^+(\tau)$$

Where $t^\pm$ are defined via

$$\frac{dt^\pm}{d\tau} = \frac{1}{1 - r_S^\pm/R}\sqrt{1 - \frac{r_S^\pm}{R} + (dR/d\tau)^2}$$

The induced metric and coordinates are as below

$$ds^2_\Sigma = -d\tau^2 + R^2(\tau)d\Omega_2^2$$

The normal form on each side is

$$n^\pm_\mu = (-\frac{dR}{d\tau}, \frac{dt^\pm}{d\tau}, 0, 0)$$

And the tangent vectors are

$$e^\mu_\tau = (\frac{dt^\pm}{d\tau}, \frac{dR}{d\tau}, 0, 0) \quad ; \quad e^\mu_\theta = (0,0,1,0) \quad ; \quad e^\mu_\varphi = (0,0,0,1)$$

Finding the angular components of the extrinsic curvature is not difficult

$$^{\pm}K_{\theta\theta} = \nabla_\theta n_\theta = R\sqrt{1 + (dR/d\tau)^2 - r_S^\pm/R}$$

$$^{\pm}K_{\varphi\varphi} = {}^{\pm}K_{\theta\theta}\sin^2\theta$$

The $\tau\tau$ component is way more cumbersome

$$K_{\tau\tau} = e^\mu_\tau e^\nu_\tau \nabla_\mu n_\nu = e^\mu_\tau \partial_\tau n_\mu - \Gamma^\alpha_{\mu\nu}e^\mu_\tau e^\nu_\tau n_\alpha = \frac{dR}{d\tau}\frac{d^2t}{d\tau^2} - \frac{d^2R}{d\tau^2}\frac{dt}{d\tau} + \frac{3r_S}{2R(R-r_S)}(\frac{dR}{d\tau})^2\frac{dt}{d\tau} - \frac{r_S(R-r_S)}{2R^3}(\frac{dt}{d\tau})^3$$

In any case, the density and pressure are given by

$$\sigma = \frac{-1}{R^2}[K_{\theta\theta}] \quad ; \quad p = \frac{1}{2}(\frac{1}{R^2}[K_{\theta\theta}] - [K_{\tau\tau}])$$

And that means we need to prove

$$\frac{d[K_{\theta\theta}]/d\tau}{[K_{\theta\theta}]} - \frac{dR}{Rd\tau} = -R\frac{dR}{d\tau}\frac{[K_{\tau\tau}]}{[K_{\theta\theta}]}$$

b) Let

$$\alpha_\pm \equiv \arcsin\frac{r_S^\pm}{R}$$

Then

$$\boxed{\sigma = \frac{1}{R}(\cos\theta_- - \cos\theta_+)\ > 0}$$

$$\boxed{p = \frac{1}{4R}(2\cos\theta_+ + \tan\theta_+ - 2\cos\theta_- - \tan\theta_-)\ > 0}$$

8

9

Chapter 4

1

a) The EL equations are

$$\frac{\partial\mathcal{L}}{\partial A_\alpha} = \nabla_\beta\frac{\partial\mathcal{L}}{\partial\nabla_\beta A_\alpha}$$

Or

$$0 = -\frac{1}{2}\nabla_\beta\left(F^{\mu\nu}\frac{\partial F_{\mu\nu}}{\partial\nabla_\beta A_\alpha}\right) = -\frac{1}{2}\nabla_\alpha F^{\mu\nu}(\delta^\beta_\mu\delta^\alpha_\nu - \delta^\beta_\nu\delta^\alpha_\mu) = \nabla_\beta F^{\alpha\beta} \quad \blacksquare$$

b)

$$T_{\alpha\beta} = g_{\alpha\beta}\mathcal{L} - 2\frac{\partial\mathcal{L}}{\partial g^{\alpha\beta}} = \boxed{F_{\alpha\mu}F_\beta{}^\mu - \frac{1}{4}g_{\alpha\beta}F^{\mu\nu}F_{\mu\nu}}$$

2

a) The action is

$$S = -m\int d\lambda\sqrt{-g_{\alpha\beta}\dot{z}^\alpha\dot{z}^\beta}$$

Then, the stress-energy tensor is

$$T_{\mu\nu}(x) = \frac{-2}{\sqrt{-g(x)}}\frac{\delta S}{\delta g^{\mu\nu}(x)} = \frac{m}{\sqrt{-g(x)}}\int d\lambda\frac{1}{\sqrt{-\dot{z}_\alpha\dot{z}^\alpha}}\dot{z}_\mu\dot{z}_\nu\delta(z^\gamma - x^\gamma)$$

This is best re-written in terms of the 4 velocity of the particle as

$$\boxed{T^{\mu\nu} = m\int d\tau\, V^\mu V^\nu\,\delta(z,x)}$$

b) The conservation is equivalent to

$$\int dx\sqrt{-g}\, A_\beta \nabla_\alpha T^{\alpha\beta} = 0$$

where A_β is any localized vector field. For a single particle, this is

$$\begin{aligned}
0 = \int dx\sqrt{-g}\, A_\beta \nabla_\alpha T^{\alpha\beta} &= m\int dx\sqrt{-g}\, d\tau\, A_\beta(x) V^\beta(\tau) V^\alpha(\tau) \nabla_\alpha \delta(z,x) \\
= -m\int d\tau\, V^\alpha V^\beta \int dx\sqrt{-g}\, \delta(z,x)\nabla_\alpha A_\beta &= -m\int d\tau\, V^\alpha V^\beta \nabla_\alpha A_\beta \\
&= -m\int d\tau\, V^\alpha \nabla_\alpha (V^\beta A_\beta) + m\int d\tau\, A_\beta V^\alpha \nabla_\alpha V^\beta \\
&= -m\langle V, A\rangle\Big|_{\tau=-\infty}^{\tau=+\infty} + m\int d\tau\, A_\beta V^\alpha \nabla_\alpha V^\beta \\
&= m\int d\tau\, A_\beta V^\alpha \nabla_\alpha V^\beta
\end{aligned}$$

Which is equivalent to the geodesic equation.

c)

3

Let's use the units in which $r_S = 1$. The bulk action is zero since this is a vacuum solution. The extrinsic curvature on the Σ_{t_i} are zero since the normals are killing fields. The non dynamical terms also cancel on the Σ_{t_i} by virtue of symmetry. Therefore the action is

$$S = 2\pi(t_2 - t_1) r^2 (K_r - K_0)\Big|_\rho^R$$

Where

$$K_r - K_0 = \frac{1}{2r^2\sqrt{1-1/r}} + \frac{2\sqrt{1-1/r}}{r} - \frac{2}{r}$$

This then gives

$$\boxed{S(R,\rho,t_1,t_2) = \pi(t_2-t_1)\big[\frac{1}{\sqrt{1-1/r}} - 4r(1-\sqrt{1-1/r})\big]\Big|_\rho^R}$$

and

$$\boxed{\lim_{R\to\infty} S(R,\rho,t_1,t_2) = \pi(t_2-t_1)\big[-1 - \frac{1}{\sqrt{1-1/\rho}} + 4\rho(1-\sqrt{1-1/\rho})\big]}$$

Chapter 5